Life-Cycle Portfolio Choices and Heterogeneous Stock Market Expectations*

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Abstract

Survey measurements of households' expectations about U.S. equity returns show substantial heterogeneity and large departures from the historical distribution of actual returns. The average household perceives a lower probability of positive returns and a greater probability of extreme returns than history has exhibited. I build a life-cycle model of saving and portfolio choices that incorporates beliefs estimated to match these survey measurements of expectations. This modification enables the model to greatly reduce a tension in the literature in which models that have aimed to match risky portfolio investment choices by age have required much higher estimates of the coefficient of relative risk aversion than models that have aimed to match wealth age-profiles. The tension is reduced because beliefs that are more pessimistic than the historical experience reduce people's willingness to invest in stocks.

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1. Introduction

When a canonical model of portfolio choice over the life cycle is calibrated to reproduce the fact that most households do not invest most of their wealth in equities, coefficients of relative risk aversion exceeding 10 are usually required.[1] In contrast, the literature on consumption and saving over the life cycle has found that coefficients of relative risk aversion of 2 or less are able to fit the data comfortably when labor income uncertainty is calibrated to match facts from widely available sources of micro data.[2] This discrepancy generates difficulties for studies that attempt to simultaneously reproduce both groups of facts using life-cycle models. Virtually all of such studies calibrate household beliefs about equity returns to match the statistical properties of actual realized returns over long periods of history.[3] The thesis of this paper is that the ability of these models to simultaneously replicate portfolio allocations and savings dramatically improves, and the tension in their parameter estimates is greatly reduced, if they are calibrated using survey measurements of consumers' actual expectations instead of historical data.

To evaluate this proposition, I use a model with two main blocks. The first is a measurement system based on Ameriks, Kézdi, et al. (2020) that I use to infer the distribution of beliefs about equity returns across U.S. households from survey measurements in the Health and Retirement Study (HRS). The second is a life-cycle model of saving and portfolio choices that builds on the workhorse model by Cocco, Gomes, and Maenhout (2005). This model adds a monetary cost of entering the stock market, which is common in the literature, and a proportional tax on stock sales that represents early-withdrawal penalties from retirement plans. It also incorporates a bequest motive and age-varying medical expense risks, important for replicating late-in-life savings.[4] I estimate the life-cycle model targeting the age profiles of savings and portfolios in the U.S. and compare the results using beliefs from survey measurements with those obtained using the standard beliefs based on historical data. The fit of the model calibrated with survey measurements of expectations is dramatically better, and its parameter estimates are much more plausible. For college graduates, using the beliefs from survey measurements reduces the distance

[1]The canonical life-cycle model of portfolio choice is due to Cocco, Gomes, and Maenhout (2005). While the purpose of that paper is not to reproduce empirical portfolio shares, it shows that, even with a coefficient of relative risk aversion of 10 and a conservative equity premium, the model-implied portfolio shares are higher than those empirically observed. The high estimated coefficients of relative risk aversion can be seen, for instance, in Fagereng, Gottlieb, and Guiso (2017), Catherine (2021) and this paper, all of which compare their full models with baseline specifications that build on Cocco, Gomes, and Maenhout (2005).

[2]See, for instance, Carroll (1997), Attanasio et al. (1999), and Gourinchas and Parker (2002).

[3]As measured, for example, by Mehra and Prescott (1985) and Shiller (1990) since the late XIX century.

[4]See. e.g., De Nardi, French, and Jones (2010) and Ameriks, Briggs, et al. (2020).

between model-implied and empirical moments from the Survey of Consumer Finances (SCF) by 46 percent[5] and reduces the estimated coefficient of relative risk aversion from 11.4 to 5.1. The high risk aversion required in the baseline calibration produces extremely high precautionary savings which, to match observed wealth, must be offset with an implausibly low time-discount factor of 0.6—suggesting that consumers discount future utility by 40 percent per year. With the beliefs from survey measurements, the discount factor increases to 0.9. Finally, the estimated monetary cost of entering the stock market falls from 1.0 percent of annual income to $ 0.[6]

Several features of the beliefs estimated from survey measurements contribute to the improved performance of the life-cycle model. The estimates imply that only 60 percent of high-school graduates and 72 percent of college graduates think there is any equity premium at all, helping to rationalize the limited rates of stock ownership and their education gaps and lowering the estimated monetary costs of entry. Among stock owners, the expected risk-adjusted returns are on average 13% and 18% lower than those implied by historical calibrations for high-school and college graduates respectively, allowing the model to match moderate portfolio shares with lower risk aversion.[7] The consequent weakening of the precautionary motive lets the model replicate savings with more patient discounting of future utility. Finally, heterogeneity in the beliefs of stock owners generates considerable dispersion in their portfolio shares, which is an empirical fact particularly difficult to reproduce using historical calibrations of beliefs.

Among the possible ways to reduce the difficulties in modeling portfolios and savings, beliefs about returns have the virtue of being susceptible of estimation from the individual-level measurements of expectations that a growing number of household surveys now include. Manski (2018), Caplin (2021), and Almås, Attanasio, and Jervis (2023) recommend the use of this type of measurements to resolve the challenge of separately identifying preferences and beliefs from observed choices, which traditional portfolio-choice models circumvent by assuming that beliefs match historical data. The measurements also produce additional empirical facts against which models can be tested. Of particular importance among these facts is that measured expectations predict portfolio allocations, a reality demonstrated by a vast literature and corroborated by this paper.

Because the estimated beliefs differ from the historical experience, individuals with

[5] Measured by the objective function of the method of simulated moments minimized in estimation. See Section 4 for details.

[6] For high-school graduates, the distance to empirical moments falls by 75 percent, the coefficient of relative risk aversion falls from 8.6 to 4.2, the annual preference time-discount factor raises from 0.3 to 0.8, and the monetary cost of entering the stock market falls from 3.1 to 2.5 percent of annual income.

[7] These figures refer to the average Sharpe ratio implied by the estimated beliefs of those who think there is an equity premium, compared to the historical Sharpe ratio of the S&P 500 index.

those beliefs would suffer welfare shortfalls if equities continued to perform as they have in the past. The model shows that these individual welfare shortfalls, quantified as a share of permanent income, would be large. They follow a hump shape across the life cycle, starting at less than 2 percent at age 24 and peaking in the years leading up to retirement at averages of 3.6 percent for high-school graduates and 7.2 percent for college graduates. Those who do not own stocks due to their more pessimistic beliefs suffer the greatest welfare shortfalls, reaching a median of 4.1 percent for high-school graduates and 7.6 percent for college graduates at the age of retirement. I analyze the variation of these welfare shortfalls across individuals and age groups and relate the model's predictions to findings in the financial literacy literature (Lusardi and Mitchell 2023).

Related literature and contributions

This paper relates and contributes to various groups of studies in household-finance and behavioral macroeconomics.

The first group of related studies explores the reasons for the discrepancies between the actual portfolio choices made by households throughout their lives and the predictions of theoretical models like Merton (1969), Samuelson (1969), Viceira (2001), and Cocco, Gomes, and Maenhout (2005). Since the detection of these discrepancies, numerous studies have attempted to address them by adding various features to their models including: more flexible specifications of households' preferences (Gomes and Michaelides 2003, 2005; Wachter and Yogo 2010; Calvet et al. 2021), richer models of labor income and its risks (Chang, Hong, and Karabarbounis 2018; Catherine 2021), and the addition of different costs that could be associated with stock ownership (Khorunzhina 2013; Campanale, Fugazza, and Gomes 2015; Fagereng, Gottlieb, and Guiso 2017). My paper contributes to this literature, showing that if beliefs are aligned with survey measurements, the predictions of the model come closer to the actual choices of households. This force can complement the mechanisms identified in this group of papers; for example, my estimates show that entry costs and rebalancing frictions become more powerful when households expect the lower risk-adjusted returns that their responses imply.

This study relates to a second group of papers concerning analyses of survey measurements of expectations about future stock market returns.[8] In this literature, a large body of work has demonstrated that expectations are heterogeneous across people and that differences in expectations are predictive of portfolio choices.[9] These facts have been

[8]See Hurd (2009) and Manski (2018) for reviews on the measurement of economic expectations in surveys.

[9]See, e.g.:Dominitz and Manski (2007), Hurd, Van Rooij, and Winter (2011), Amromin and Sharpe (2014), Drerup, Enke, and von Gaudecker (2017), Ameriks, Kézdi, et al. (2020), Giglio et al. (2021), and Calvo-Pardo,

corroborated in multiple surveys, samples, and countries, as well as by using different ways of eliciting expectations. In addition to their predictive power, other important features of measured expectations about stock returns, such as pessimism (Dominitz and Manski 2007; Hurd, Van Rooij, and Winter 2011), socioeconomic gradients (Das, Kuhnen, and Nagel 2020), and rounding (Manski and Molinari 2010; Giustinelli, Manski, and Molinari 2022) have been established. A critical characteristic for modeling these measured expectations is that most of their variation comes from individual "fixed effects," cross sectional differences that persist over time. Only a small part of the variation of fixed effects across individuals is explained by sociodemographic characteristics (Giglio et al. 2021). My contribution to this literature is a model that I use to estimate an interpretable representation of the persistent component of beliefs (structural analogues to individual fixed effects) that can be incorporated into life-cycle portfolio-choice models. The model, which builds on Kézdi and Willis (2011), Ameriks, Kézdi, et al. (2020), and Giustinelli, Manski, and Molinari (2022), accounts for persistent and heterogeneous rounding patterns, and its estimates capture many of the empirical features of beliefs that have been highlighted in the literature.

The formation and dynamics of expectations is an important area of study that this paper does not address. It is a well established fact in this domain that experiences—recent and distant, personal and vicarious—have an effect on expectations (see, e.g., Malmendier and Nagel 2011, 2016; Amromin and Sharpe 2014; Greenwood and Shleifer 2014; Coibion and Gorodnichenko 2015; Bailey et al. 2018; Bordalo et al. 2019). However, in spite of the robustness of this fact and its macroeconomic significance, experiences and common belief revisions (time fixed-effects) capture only a small share of the micro-level variation in measured expectations about stock returns (Giglio et al. 2021). Therefore, with the goal of modeling households' individual choices, this paper focuses instead on the persistent components of individual expectations, which capture around half of their variation.[10]

This paper also relates to a growing literature that uses measurements of individual expectations in the estimation of structural economic models.[11] Studies such as Guiso, Jappelli, and Terlizzese (1992), Dominitz and Manski (1997), Lusardi (1997, 1998), and Caplin et al. (2023) examine measurements of households' expectations about their in-

Oliver, and Arrondel (2022).

[10] Campanale (2011), Peijnenburg (2018), and Foltyn (2020) have analyzed portfolio-choice models in which individuals learn about the distribution of risky returns form their experiences. While learning, coupled with participation costs, helps to replicate participation patterns, it does not amend the basic model's prediction about conditional portfolio shares. Therefore, some of the studies rely on additional mechanisms like ambiguity aversion.

[11] See Koşar and O'Dea (2023) for an excellent review of this literature and Manski (2018), Caplin (2021), and Almås, Attanasio, and Jervis (2023) for arguments in favor of this approach.

come dynamics, showing that the expectations differ from standard estimates that use administrative data, and using the measured expectations in models of saving decisions and job-transitions. Similarly, measurements of beliefs are increasingly used in models of other economic decisions, such as educational and occupational choices (Arcidiacono et al. 2020; Wiswall and Zafar 2021) and parental investments (Almås, Attanasio, and Jervis 2023). However, despite the well documented differences between households' measured expectations and the standard historically-based calibrations, this paper is the first to use survey measurements of expectations about equity returns in a life-cycle model to explain the savings and portfolio choices of U.S. households to the best of my knowledge.

Finally, this paper contributes to the literature on wealth differences between socio-demographic groups. Studies in this literature have identified cross-group differences that are difficult to explain using the life-cycle/permanent-income models of consumption and saving. Precautionary savings, differences in time-preference rates, and the differential effects of social programs on the incentive to save have been explored as explanations for these difficulties (Carroll 1994; Hubbard, Skinner, and Zeldes 1995; Cagetti 2003). More recently, Lusardi, Michaud, and Mitchell (2017) show that cross-group differences in financial proficiency have the potential to explain a large part of the empirical relationship between savings as a fraction of income and educational attainment, and account for a large share of wealth inequality between groups with different levels of education. I add to this literature by demonstrating that indeed, when a life-cycle model accounts for measurable differences in expectations about asset returns, it can replicate educational differences in wealth and portfolios with much smaller cross-group differences in preferences.

The rest of this paper is organized as follows. Section 2 presents basic empirical facts about U.S. households' wealth, stock holdings, and beliefs about future stock returns. Section 3 presents the model of beliefs and life-cycle saving and portfolio choices. Section 4 discusses my strategy for estimating the model. Section 5 presents the estimation results and discusses their implications. Section 6 quantifies the welfare losses that individuals may suffer from misspecified beliefs about future stock returns. Section 7 concludes.

2. The Portfolios and Expectations of U.S. Households

In this section, I review various empirical facts about stockholding that challenge the predictions of portfolio-choice models in which households think that future stock returns will follow their historical distribution. Then, using 16 years of measured expectations, I show that U.S. households' subjective distributions of stock returns appear to deviate substantially from the historical distribution of actual stock returns. The differences between measured expectations and the historical distribution have various features that could explain some of the discrepancies between the predictions offered by traditional models and U.S. households' actual stockholding behaviors.

2.1 Aggregate Patterns of Stockholding in the U.S.

This section examines the stock-market participation and portfolio patterns of U.S. households, focusing on the comparison between high-school and college graduates. I use the Survey of Consumer Finances to highlight patterns that deviate from the predictions of standard life-cycle portfolio-choice models. The deviations include lower-than-predicted participation rates and shares of wealth in stocks, low variation in the share of wealth in stocks across age-groups, and higher participation and wealth in stocks among wealthier individuals.

To study aggregate patterns in U.S. households' savings and stock holdings, I use the triennial *Survey of Consumer Finances* (SCF). The SCF provides a comprehensive picture of American households' balance sheets, including "summary files" with useful aggregates such as the total financial assets and stock holdings of each surveyed economic unit. These stock holdings include both assets owned directly and those owned indirectly through, e.g., mutual funds and retirement accounts. The summary files estimate indirect stock holdings based on respondents' descriptions of the types of assets that a given fund or account invests in. Although account-level data can offer more precise measurements of stock holdings (as noted by Parker et al. 2022), the advantage of the SCF lies in its comprehensive coverage of a nationally representative sample of economic units and all their financial accounts.

Table 1 presents summary statistics for the main variables of interest, as well as demographic variables that describe the sample. The statistics are shown for the full set of observations and split by the respondent's highest level of education. The average respondent is 51 years old. Out of respondents, 13% lack a high-school degree, 55% have a high-school degree but no college degree, and 32% have obtained a college degree. Due to changes in educational access, those without a high-school degree were born, on average, ten years earlier (1945) than those with high-school or college degrees (1955 and 1957).

My analysis in this paper focuses on high-school and college graduates only. There are three main reasons for the choice to exclude those without a high-school degree. First, their number of available observations is much lower than that of high-school and college graduates. When grouped into age-bins, as my analysis requires, the number of

Table 1: Summary statistics: main variables of interest

	All		Less than H.S.		High School		College	
Variable	Mean	St. Dev.	Mean	St. Dev.	Mean	St. Dev.	Mean	St. Dev.
Age and education								
Birth Year	1,954.46	18.99	1,944.82	21.47	1,955.50	18.58	1,956.56	17.41
Age	50.82	17.18	58.56	18.33	49.73	17.15	49.57	15.91
Less Than H.S.	0.13	0.33	—	—	—	—	—	—
High-School	0.55	0.50	—	—	—	—	—	—
College	0.32	0.47	—	—	—	—	—	—
Income, wealth, and stock ownership								
Income (1000s)	66.20	139.89	30.73	46.01	50.97	60.03	107.10	227.77
Fin. Assets (1000s)	229.35	1,571.76	42.45	301.80	105.53	744.53	520.81	2,578.26
Owns stocks?	0.51	0.50	0.19	0.39	0.46	0.50	0.73	0.44
Stocks/Fin. Assets	0.23	0.30	0.08	0.22	0.20	0.29	0.35	0.32
Cond. stock share	0.45	0.29	0.43	0.31	0.43	0.29	0.47	0.28

The summary statistics in this table come from pooling the observations from the 1989 to 2019 SCF waves. The sample is restricted to respondents above the age of 21 with non-negative financial wealth and a stock-share of financial wealth between 0 and 100%. All calculations use pooled survey weights. The unit of analysis in SCF it is the "primary economic unit". Wealth and income are expressed in 2010 U.S. dollars and were adjusted using the using the CPI index. I refer to individuals that do not posses a high-school diploma or GED as "Less than H.S. " to those with a high-school diploma or GED but no college degree as "High school" and to those with a college degree as "College."

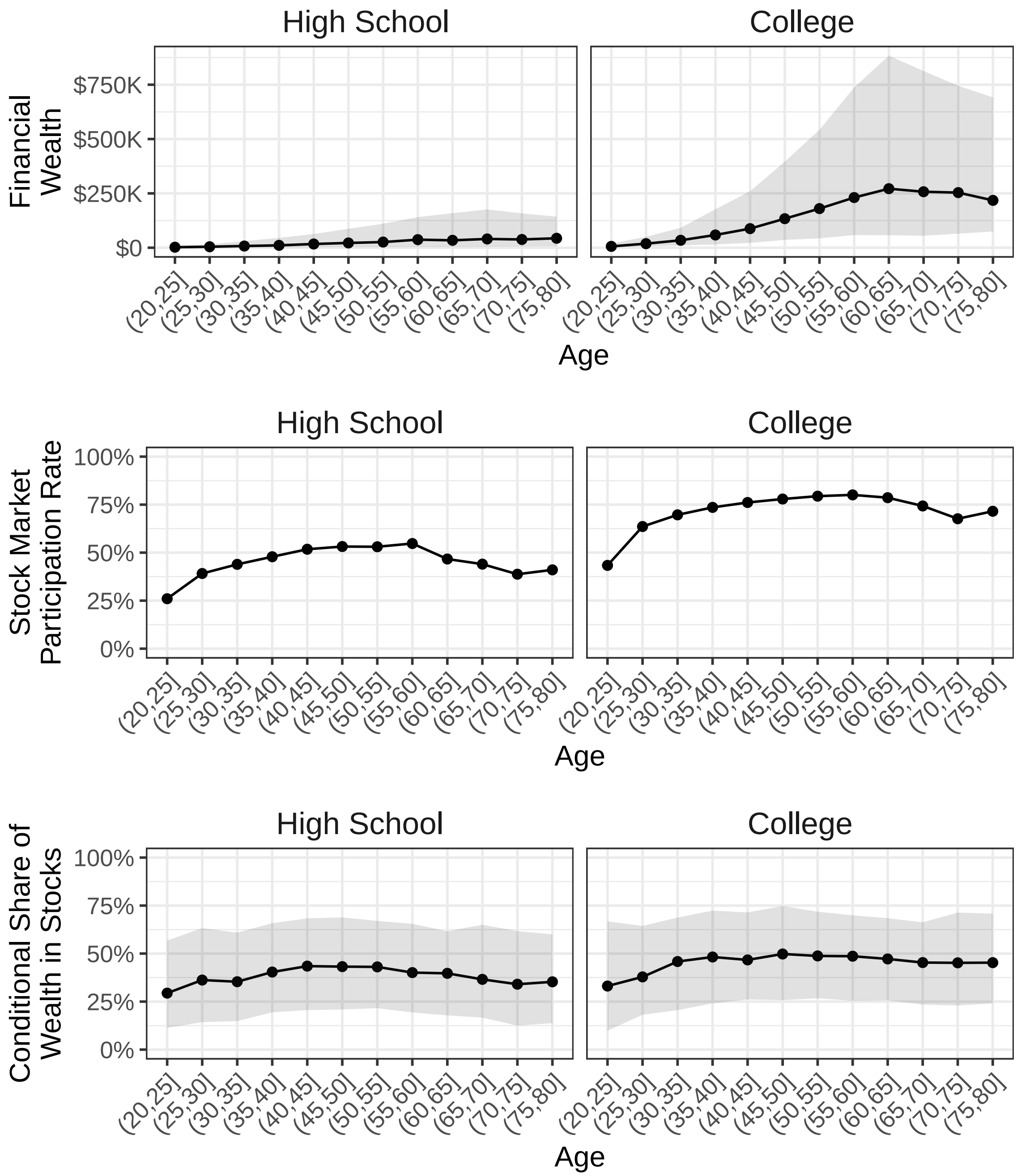
observations per group becomes too low to produce sufficiently precise estimates of the moments of interest. Second, as will be discussed below, a large fraction of respondents without a high-school degree answer "does not know/refuse" to probabilistic questions in the *Health and Retirement Study* upon which my analysis relies. Third, as Table 1 shows, households in which the respondent does not have a high-school degree have low incomes and levels of wealth. As argued by Hubbard, Skinner, and Zeldes (1995) the saving decisions of these households are severely impacted by social programs that are absent from the modeling exercise in this paper.

Table 1 reproduces the low rates of stockholding that constitute the "stockholding puzzle" (Haliassos and Bertaut 1995). The table shows that, out of all observed economic units, only 51% owns stocks directly or indirectly. This fact is at odds with the prescription that all households should own stocks, which is a feature of frictionless models in which everyone expects the stock market to perform in the future as it has historically. There is a positive correlation between level of education and stock market participation, with the lowest participation rate (19%) observed among individuals with less than a high-school degree and the highest participation rate (73%) observed among individuals with a college degree. Multiple factors could contribute to this correlation. One explanation could be that individuals with higher education are more likely to believe in the existence of an equity premium. Another possibility could be the existence of barriers that disproportionately affect those with lower education; for instance, monetary participation fees could limit the participation of those with lower education more severely, as they have lower incomes and wealth on average.

The shares of wealth in stocks among those who participate in the stock market shows a weaker relationship with education than the rate of participation, but they are also lower than what baseline models predict. Given the low rates of participation, the mean share of wealth in stocks is low, at 23%. A more meaningful measure is the share of wealth invested in stocks among those who participate, commonly referred to as the conditional share of wealth in stocks. College graduates have a slightly higher average conditional share of 47% compared to that of high-school graduates, which is 43%. These levels are low compared to the predictions made by life-cycle portfolio-choice models, like that of Cocco, Gomes, and Maenhout (2005) which, even with a relative-risk aversion coefficient of 10 and a moderate equity premium, predicts a share of wealth in stocks ranging from 60 to 100 percent depending on the agent's age.

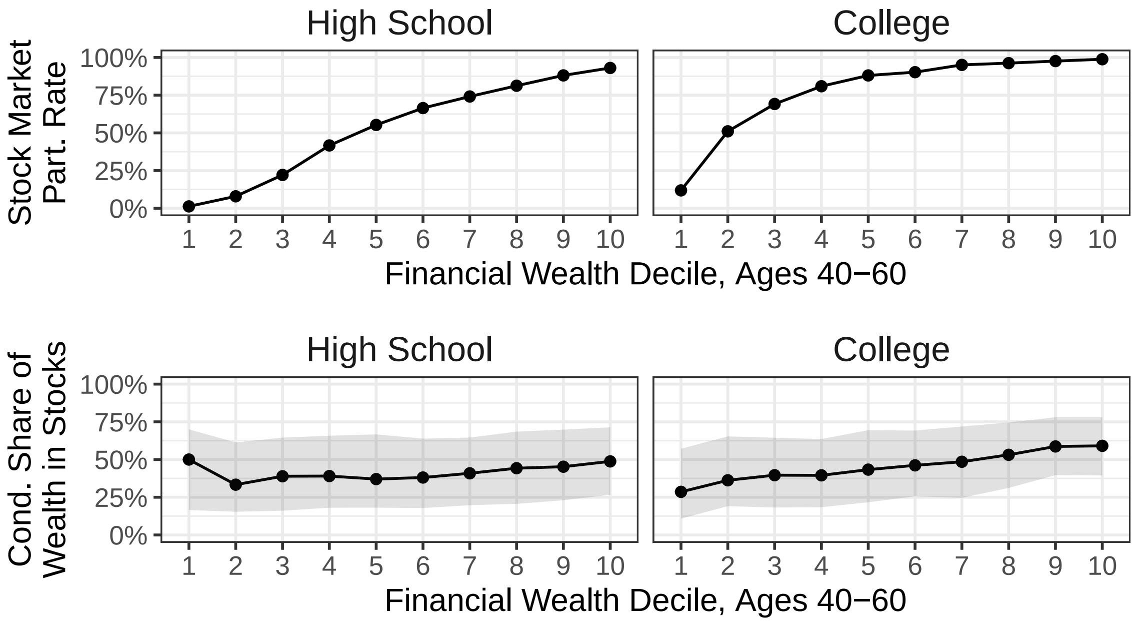
The age-patterns of stock market participation and conditional wealth in stocks are similar for high-school and college graduates. Figure 1 shows statistics on financial wealth, stock-market participation rates, and conditional shares calculated on five-year age-bins. The figure shows that, in spite of differences in levels, the stock market participation rates of both high-school and college graduates follow a similar inverted "U" shape as they age. For both groups, the highest participation rates occur between the ages of 56 to 60, with 55% for high-school graduates and 80% for college graduates. In contrast, the conditional share of wealth in stocks remains relatively stable across different age bins, showing little variation in the 25th, 50th, and 75th percentiles of its distributions. In every age group, the distributions of the conditional share of wealth in stocks for high-school and college graduates are similar to each other.

The stability of conditional shares of wealth in stocks over different age bins in the SCF is inconsistent with traditional portfolio-choice models, which prescribe that these shares must decline with age. These models derive this prescription from the assumption that a person's expected future lifetime earnings—their "*human wealth*"—act as a hedge against stock-market fluctuations. Therefore, it is optimal for a young person with high human wealth to allocate most of his investable wealth to stocks, and to reduce his exposure as he ages and his human wealth decreases. Indeed, in Cocco, Gomes, and Maenhout's (2005) benchmark calibration, young agents invest 100 percent of their wealth in stocks and gradually lower this share as they age until around 60 percent. Parker et al. (2022) show that the increasing popularity of target-date funds has brought the conditional shares of



The summary statistics in this table come from pooling the observations from the 1989 to 2019 SCF waves and grouping them into 5-year age bins. For wealth and the conditional share of wealth in stocks, the solid line and points display the median, and the shaded areas span from the 25th to the 75th percentile. For the participation rate, the solid line and points displays the fraction of stock market participants. The sample is restricted to respondents with non-negative financial wealth and a stock-share of financial wealth between 0 and 100%. All calculations use pooled survey weights. Wealth is expressed in 2010 U.S. dollars and was adjusted using the using the CPI index.

Figure 1: Wealth and stockholding over the life cycle



The summary statistics in this table come from pooling the observations with respondents aged 40 to 60 from the 1989 to 2019 SCF waves and grouping them into wealth deciles. For conditional share of wealth in stocks, the solid line and points display the median, and the shaded areas span from the 25th to the 75th percentile. For the participation rate, the solid line and points displays the fraction of stock market participants. The sample is restricted to respondents with non-negative financial wealth and a stock-share of financial wealth between 0 and 100%. All calculations use pooled survey weights.

Figure 2: Stockholding across the wealth distribution

recent cohorts more in line with the declining patterns prescribed by life-cycle models.

The relationship between stock holdings and wealth observed in the SCF also deviates from the predictions of traditional portfolio models. In those models, a higher level of investable wealth makes human wealth with its hedging role a smaller fraction of an individual's resources, thus making it optimal for the individual to lower his share of investable wealth in stocks. However, Figure 2 shows that this relationship does not hold in the SCF data. On the contrary, for both high-school and college graduates the median conditional share of wealth in stocks increases modestly with increasing wealth.[12] Stock market participation increases sharply with wealth, ranging from 1% and 12% in the first wealth decile to 93% and 99% in the last wealth decile for high-school and college graduates, respectively. These participation patterns also pose a challenge for traditional models, as the frequent assumption of a financial participation cost fails to explain why some households do not own stocks despite having considerable wealth or why some households do not enter the stock market after large wealth windfalls (Briggs

[12]For high-school graduates, the conditional share has a noticeable decrease from the first to the second wealth decile. This discrepancy may arise from extreme selection, as almost no individual in the lowest wealth decile holding only a high-school diploma owns stocks.

et al. 2021). Foltyn (2020) shows that experience-based learning about returns and under-diversification can bring the model-implied relationship between wealth and participation closer to the observed data.

In sum, while there are significant differences in the stock-market participation rates of high-school and college graduates, the shares of wealth in stocks of those who participate are comparable. Traditional life-cycle portfolio-choice models have difficulties in explaining the participation rates and conditional stock shares of these two groups. Additionally, data from the SCF does not support these model's predictions regarding the relationships between these variables and age and financial wealth. I now investigate U.S. households' measured beliefs about stock returns as a potential explanation for these observed patterns.

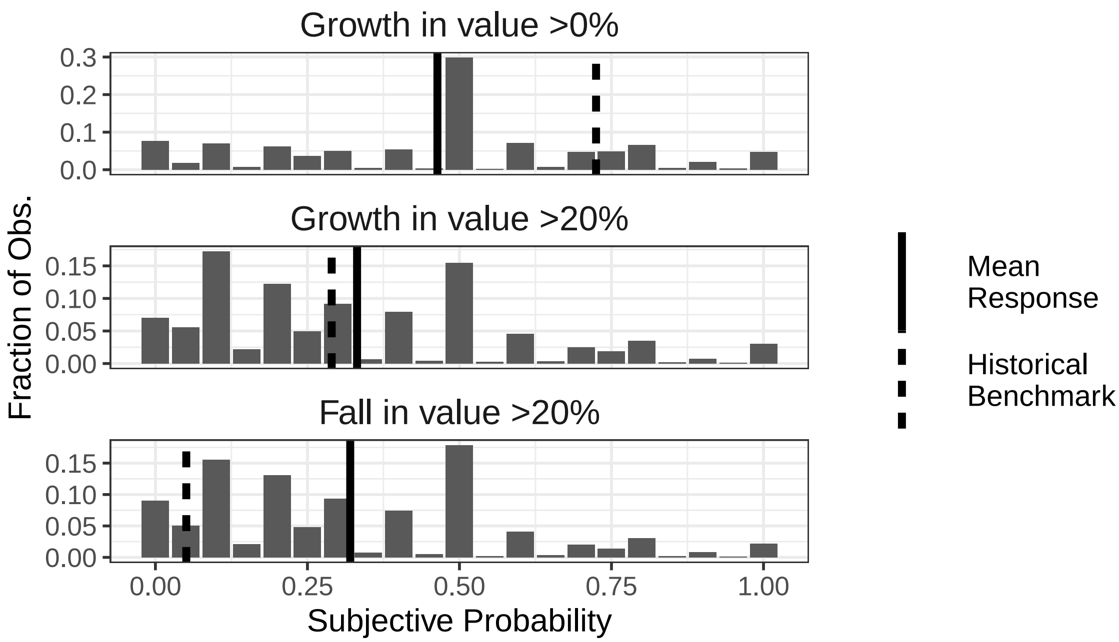
2.2 U.S. Households' Expectations About Stock-Returns

Survey measurements of people's expectations about the future performance of the stock market vary substantially across individuals and deviate from historical benchmarks. Compared to the historical experience, the average person underestimates the probability of positive returns and overestimates the probability of extreme returns (positive and negative), with the magnitude of these differences varying systematically with the individual's education level. A large fraction of the variation in these survey measurements corresponds to persistent heterogeneity in people's expectations, and this heterogeneity robustly associates with differences in people's stockholding behavior.

To characterize U.S. households' perceptions about the future performance of the stock market, I use the *Health and Retirement Study* (HRS). The HRS is a biennial longitudinal survey of U.S. adults over the age of 50 that gathers detailed information on respondents' health, financial status, employment, and expectations. Since 2002, the expectations module of the HRS has included questions about the future performance of the stock market. My analysis uses the following questions regarding expectations:

- [$P^{\geq 0}$] *"By next year at this time, what is the percent chance that mutual fund shares invested in blue chip stocks like those in the Dow Jones Industrial Average will **be worth more** than they are today?"*

- [$P^{\geq 20}$] *"By next year at this time, what is the percent chance that mutual fund shares invested in blue-chip stocks like those in the Dow Jones Industrial Average will **have gained in value by more than 20 percent** compared to what they are worth today?"*



Responses are rounded to the nearest multiple of 5% and each bar reports the fraction of observations corresponding to each multiple. The sample consists of individuals above the age of 50 who report being the financial respondent of the household. The question about positive growth in value (first row) was added in 2002 and the other two were added in 2008; therefore, the samples of both columns do not exactly match. The "historical benchmark" lines correspond to the fraction of years between 1881 and 2018 that the S&P500 index had returns higher that 0%, higher than 20%, and lower than -20%; these calculations are based on the accompanying data file to Chapter 26 of Shiller (1990).

Figure 3: Probabilistic assessments about stock returns

- [$P^{\leq -20}$] *"By next year at this time, what is the percent chance that mutual fund shares invested in blue-chip stocks like those in the Dow Jones Industrial Average will have* ***fallen in value by more than 20 percent*** *compared to what they are worth today?"*

I use $P^{\geq 0}$, $P^{\geq 20}$, and $P^{\leq -20}$ to denote these measurements. The HRS first measured $P^{\geq 0}$ in 2002, with $P^{\geq 20}$ and $P^{\leq -20}$ following in 2008.[13]

Responses to the questions about future stock returns are disperse and their averages deviate considerably from the historical performance of the stock market. Figure 3 depicts the distribution of $P^{\geq 0}$, $P^{\geq 20}$, and $P^{\leq -20}$ across all survey waves and shows that—far from concentrating around estimated answers from a common subjective distribution of returns—the responses span wide ranges without signs of agreement. I compare the responses to annual returns of the S&P 500 index from 1881 to 2018, as reported by Shiller (1990). During this period, the S&P 500 saw positive returns on 72% of the years, returns

[13]The 2008 wave asked various different combinations of "gain/fall in value by X%" to different individuals. The "gain/fall in value by 20%" versions of the question were incorporated in 2010. I use the individuals who drew $X = 20$ in 2008 to construct $P^{\geq 20}$ and $P^{\leq -20}$ for that wave.

Table 2: Probabilistic assessments about stock returns and education

Question	Mean	St. Dev.	N. Obs	Fract. DK/RF
Less than High School				
$P^{\geq 0}$	0.40	0.30	19,175	0.29
$P^{\geq 20}$	0.38	0.30	5,136	0.05
$P^{\leq -20}$	0.30	0.27	5,401	0.04
High School				
$P^{\geq 0}$	0.45	0.26	59,273	0.14
$P^{\geq 20}$	0.34	0.25	21,341	0.03
$P^{\leq -20}$	0.33	0.25	21,380	0.02
College				
$P^{\geq 0}$	0.53	0.25	23,257	0.06
$P^{\geq 20}$	0.30	0.23	10,594	0.01
$P^{\leq -20}$	0.31	0.21	10,391	0.01

The sample consists of individuals above the age of 50 who report being the financial respondent of the household. The question about positive growth in value (first row) was added in 2002 and the other two were added in 2008; therefore, the samples for the questions do not exactly match. Furthermore, each wave the questions $P^{\geq 20}$ and $P^{\leq -20}$ are asked only to participants who do not answer "does not know/refused" to $P^{\geq 0}$.

greater than 20% on 29% of the years, and returns below −20% on 5% of the years. The average response for the probability of positive nominal returns, which is 46%, is 26 percentage points below its historical benchmark. Conversely, the average responses for the probabilities of extreme returns, which were 33% for $P^{\geq 20}$ and 32% for $P^{\leq -20}$, exceed their respective historical benchmarks by 4 and 27 percentage points. The deviations of average responses from historical benchmarks and households' pessimism about the chances of positive returns in particular are well known facts that have found support across multiple surveys, conducted in the U.S. and abroad (see Hurd 2009; Manski 2018, for reviews).

Expectations about future stock returns show a systematic relationship with educational attainment yet significant variability remains among individuals with the same level of education. Table 2 presents summary statistics of $P^{\geq 0}$, $P^{\geq 20}$, and $P^{\leq -20}$ for respondents with different levels of education. While all groups are pessimistic about the probability of positive returns ($P^{\geq 0}$), the average response increases steeply with education, from 40% for those without a high-school degree to 53% for college graduates. This pattern is consistent with the findings of past studies, which have shown that more educated individuals tend to have a more optimistic outlook on stock returns (Dominitz and Manski 2011; Hurd, Van Rooij, and Winter 2011; Das, Kuhnen, and Nagel 2020). The degree to

Table 3: Fractions of "rounded" probabilistic responses

	Fraction of Answers in Group					
Question	{0%, 100%}	50%	{25%, 75%}	Other×10%	Other×5%	Other
$P^{\geq 0}$	0.12	0.30	0.09	0.44	0.05	0.01
$P^{\geq 20}$	0.09	0.15	0.07	0.57	0.09	0.02
$P^{\leq -20}$	0.10	0.18	0.06	0.55	0.08	0.02

The sample consists of individuals above the age of 50 who report being the financial respondent of the household. The question about positive growth in value (first column) was added in 2002 and the other two were added in 2008; therefore, the samples of the three columns do not exactly match. Other×10% = {10, 20, 30, 40, 60, 70, 80, 90}%. Other×5% = {5, 15, 35, 45, 55, 65, 85, 95}%. "Other" represents answers that do not fall into any of the other groups.

which the average respondent overestimates the probability of extreme returns also varies with educational attainment, with more educated households generally giving lower responses.[14] However, within-group variability is greater than cross-group differences, as within-group standard deviations are higher than 20 percentage points for all questions.

Table 2 also shows the fraction of participants who refused to answer each of the questions or answered with "do not know." The refusal/unsure rate is much higher for $P^{\geq 0}$ than for $P^{\geq 20}$ and $P^{\leq -20}$. This disparity arises as participants who refuse to answer $P^{\geq 0}$ or answer this question with "do not know" are not asked $P^{\geq 20}$ or $P^{\leq -20}$. While the refusal/unsure rates for high-school and college graduates are moderate, they reach 29% for $P^{\geq 0}$ among individuals without a high-school degree. Such a high fraction of refusal/unsure answers casts doubt on the representativeness of respondents without a high-school degree who do answer the probabilistic questions. For this reason, in addition to those presented in Section 2.1, I limit the modeling exercises in this paper to high-school and college graduates.

Rounding is pervasive in probabilistic assessments about future stock returns. Figure 3 shows that there are large masses of answers in focal points like 0%, 50%, and 100%, and that all multiples of 10% occur more frequently than their neighboring multiples of 5%. Table 3 presents the fraction of responses that belong to different groups of frequent answers with the groups following a similar partition to the one proposed by Giustinelli, Manski, and Molinari (2022). For each question, more than 8% of the answers are multiples of 100% (0 or 100%), more that 80% are multiples of 10%, and more than 97% are multiples of 5%. Using the full expectations module of the HRS, Giustinelli, Manski, and Molinari (2022) show that individuals tend to round questions in the same domain (e.g., health or

[14]The only exception to this pattern is the average $P^{\leq -20}$ of those without a high-school degree, which is the lowest of the three groups.

Table 4: Sources of variation in probabilistic assessments

	Models' R^2		
Question	Time F.E.	Indiv. F.E.	Two-Way F.E
$P^{\geq 0}$	0.0074	0.4630	0.4694
$P^{\geq 20}$	0.0095	0.6312	0.6355
$P^{\leq -20}$	0.0107	0.5988	0.6045

The table reports R^2 statistics from regressing each measurement on individual fixed-effects, month-of-interview fixed-effects, and both. The sample consists of individuals above the age of 50 who report being the financial respondent of the household. The question about positive growth in value (first row) was added in 2002 and the other two were added in 2008, therefore the samples of the three rows do not exactly match.

finances) to consistent levels of coarseness, even though the level of rounding coarseness varies between individuals and across domains. Based on these findings, the model of beliefs that I use in this paper accounts for rounding practices that are stable over time but heterogeneous across individuals.[15]

A large fraction of the variation in probabilistic assessments about stock returns comes from cross-sectional differences between individuals that persist over time. In their analysis of the macroeconomic beliefs of Vanguard account holders, Giglio et al. (2021) show that, for all the measurements of expectations in their analysis, persistent differences in expectations across individuals (individual fixed effects) explain a vastly larger fraction of variation than common movements in expectations over time (time fixed effects). I replicate their analysis with the HRS sample, which includes both investors and-non investors and spans a longer period of time. Table 4 presents the R^2 statistics of regressions of the form:

$$\begin{aligned} \text{Indiv. F.E.,} \quad P^x_{i,t} &= a_i + \epsilon_{1,i,t} \\ \text{Time F.E.,} \quad P^x_{i,t} &= b_t + \epsilon_{2,i,t} \\ \text{Two-way F.E.,} \quad P^x_{i,t} &= a_i + b_t + \epsilon_{3,i,t}, \end{aligned}$$

where P^x is one of the probabilistic assessments ($P^{\geq 0}$, $P^{\geq 20}$, or $P^{\leq -20}$), a_i are individual fixed-effects, b_t are month-of-interview fixed effects, and $\epsilon_{\cdot,i,t}$ are time-specific idiosyncratic errors. The results are consistent with the findings of Giglio et al. (2021): individual fixed-effects explain a much larger fraction of the variance in responses than time fixed-effects. For the probability of positive returns $P^{\geq 0}$, individual fixed-effects capture 46% of

[15] Survey responses that reflect rounding and the use of heuristics are pervasive in other related contexts such as the hypothetical choice of retirement wealth allocations (Bateman et al. 2017).

Table 5: Stockholding and the subjective probability of positive returns

	Participation (LPM)			Share* (Tobit)	
	Model 1	Model 2	Model 3	Model 4	Model 5
$P^{\geq 0}$	0.26***	0.17***	0.04***	0.41***	0.19***
	(0.01)	(0.01)	(0.01)	(0.02)	(0.01)
$P^{\geq 0}$: DK/Refused	−0.10***	−0.06***	−0.01*	−0.23***	−0.11***
	(0.01)	(0.01)	(0.00)	(0.02)	(0.01)
Mean	0.29	0.29	0.29	0.18	0.18
Household F.E.			✓		
Household R.E.					✓
Age and Year Dummies	✓	✓	✓	✓	✓
Education Dummies		✓		✓	✓
Log-Income (HH)		✓	✓	✓	✓
Gender		✓		✓	✓
Num. Obs	78062	78062	78062	78062	78062
R^2	0.06	0.13	0.02		
Adj. R^2	0.06	0.13	−0.30		

$^{***}p < 0.001$; $^{**}p < 0.01$; $^{*}p < 0.05$

Participation is a binary indicator for owning stocks outside of IRA/Keogh accounts. Share is the share of non-IRA/Keogh financial wealth that is represented by stocks. Non-IRA/Keogh financial wealth includes checking and saving accounts, certificated of deposit, bonds, and stocks outside of retirement accounts. Both variables are measured at the household level. The sample consists of individuals above the age of 50 who report being the financial respondent of the household. For the linear-probability models, errors are clustered at the respondent level.

the variance. For the probabilities of extreme returns $P^{\geq 20}$ and $P^{\leq -20}$, they capture 63% and 59% of the variance, respectively. Time fixed-effects, in contrast, capture no more than 1.5% of the variance in any of the questions. As Giglio et al. (2021) point out, the persistent cross-sectional heterogeneity in beliefs manifest in these measurements stands at odds with many of the models used in macroeconomics and finance.

Previous studies have consistently shown that cross sectional differences in individuals' beliefs about stock returns predict their stock-holding behavior.[16] I corroborate this finding using the HRS sample. Table 5 presents results from regressing measures of a household stock ownership outside of retirement accounts on subjective probabilities of positive returns.[17] The first three columns regress a binary indicator of stock ownership on $P^{\geq 0}$

[16] Dominitz and Manski (2007), Hurd, Van Rooij, and Winter (2011), Amromin and Sharpe (2014), Drerup, Enke, and von Gaudecker (2017), Ameriks, Kézdi, et al. (2020), Giglio et al. (2021), and Calvo-Pardo, Oliver, and Arrondel (2022) are some examples. See Hurd (2009) and Manski (2018) for reviews.

[17] The measures of households' balance sheets that I use come from the RAND HRS longitudinal file, which homogenizes various variables across survey waves. I limit this motivating analysis to assets outside

and different sets of controls. As in previous studies, the probability of positive returns predicts participation, and controlling for socioeconomic characteristics like education and income attenuates the relationship. According to the estimates in the second column, with a subjective probability of positive returns 10 percentage points higher, a respondent is 1.7 percentage points more likely to own stocks. The fourth column uses a Tobit model with censoring at 0.0 and 1.0 to examine the relationship between $P^{\geq 0}$ and a respondent's share of wealth in stocks. The estimates suggest that a 10-percentage-point increase in a respondent's subjective probability of positive returns is associated with a 4.1-percentage-point increase in his expected share of wealth in stocks.

The within-individual relationship between variations in beliefs about returns and variations in stockholding behavior is weaker than the relationship observed across individuals. Table 5 demonstrates this fact by adding individual fixed effects to the linear model of participation in column 3 and individual random-effects to the Tobit model of the share of wealth in stocks in column 5. Adding individual fixed-effects to the linear model of participation reduces the coefficient of $P^{\geq 0}$ to less than one quarter of the estimate from the model with socioeconomic controls in column 2. In the Tobit model of share of wealth in stocks, individual random-effects reduce the coefficient of $P^{\geq 0}$ to half of the estimate from the initial model in column 3. These findings align with the results of Giglio et al. (2021), who show that changes in people's beliefs about returns do not predict when they rebalance their portfolios. In sum, while individuals have persistently different beliefs about returns that predict their portfolio choices, these beliefs fluctuate, and the extent to which an individual changes his portfolio when his beliefs change is limited.

This section reviewed several challenges faced by traditional life-cycle models in explaining U.S. households' stockholding patterns. Stock-holding rates are low and correlated with education, those who own stocks do not allocate as much wealth to them as these models predict, and the relationships between stockholding, age, and wealth predicted by these models do not match those in the data. These conflicting predictions arise from models specifying agents' expectations about stocks based on the historical experience, an assumption at odds with survey measurements of these expectations. Several properties of measured expectations could help bridge the gap between portfolio choice models and U.S. households' behavior. For instance, heterogeneous beliefs correlated with education could help explain limited participation rates, and pessimism could rationalize low shares of wealth in stocks. In the next sections, I explore whether this is indeed the case, and if a life-cycle model matching expectations to survey measurements can improve

of IRA/Keogh accounts because, over time, the HRS has changed the ways in which it asks respondents to report the types of assets in which the balances of these accounts are invested.

the fit of savings and portfolio choices.

3. A Model of Beliefs and Stock-Holding

The model that I propose consists of two parts. The first is a measurement system that I use to interpret the probabilistic responses, $P^{\geq 20}$, $P^{\geq 20}$, and $P^{\leq -20}$, and to estimate the distribution of beliefs about stock returns across the population. The second part is a life-cycle consumption-saving model with portfolio allocation decisions in which agents' beliefs about asset returns as fixed and exogenous. I discuss each part in turn.

3.1 Representing the Beliefs of U.S. Households

To represent the beliefs about stock-returns of U.S. households, I construct and estimate a model that maps their probabilistic assessments to heterogeneous subjective distributions of stock-returns. The advantage of the model that I propose is that it represents beliefs in a way that can be estimated directly from survey measurements and then plugged into life-cycle models. I estimate the model using almost twenty years of longitudinal measurements of U.S. households' expectations about stock-returns. The estimates suggest that there are permanent differences in beliefs about stock-returns across people, that the average person is more pessimistic about stocks than the historical experience would suggest, and that more educated people are more optimistic about stocks. All these features are consistent with previous findings in the literature studying people's expectations about stock returns.

3.1.1 *A model of beliefs and probabilistic assessments*

The model that I propose is an adaptation of the ones used by Kézdi and Willis (2011) and Ameriks, Kézdi, et al. (2020). Every person believes that stock-returns follow a distribution that can change from one person to the next but does not change over time. People use their subjective distributions to produce probabilistic assessments, but their answers are also perturbed by time-varying shocks that represent survey errors and short-term fluctuations in their beliefs. People also round their answers to different, but personally-stable degrees, i.e. some round all their answers to the nearest multiple of 5%, others to the nearest multiple of 10%, 25%, 50%, or 100%. I identify and estimate the distribution of the persistent part of beliefs across the population using the longitudinal nature of the data and the fact that multiple questions about stock returns are asked in various survey waves.

In the model, people believe that nominal stock-returns are log-normally distributed with individual-specific parameters μ_i and σ_i:

$$\ln \tilde{R}_{t+1} \overset{i}{\sim} \mathcal{N}(\mu_i, \sigma_i).$$

The individual-specific parameters of people's beliefs are fixed over time and follow a distribution Ω across the population, $(\mu_i, \sigma_i) \sim \Omega$. The assumption that beliefs differ across people and are fixed over time is a parsimonious way of modeling the fact that most of the panel-variation in probabilistic assessments about stock-returns comes from persistent differences across individuals (Giglio et al. 2021). The literature has suggested various mechanisms that could generate this persistent heterogeneity, offering differences in lived experiences (Malmendier and Nagel 2011) or in costs of and returns to learning about stocks (Kézdi and Willis 2011) as two examples. I take belief-heterogeneity as given and model it using the distribution Ω, which I estimate.

People make probabilistic assessments about stock-returns using their log-normal beliefs, but their responses are subject to time- and question-specific disturbances and rounded to different degrees. Manski and Molinari (2010) and Giustinelli, Manski, and Molinari (2022) demonstrate that rounding is prevalent in the answers to probabilistic questions in the HRS, and that the degree or "coarseness" of rounding varies across respondents but is stable over time. These studies show that ignoring the rounding patterns present in the data and taking probabilistic assessments at face value can alter an econometric model's empirical estimates and precision. I account for these issues in my model by assuming that each person i has a "rounding-type" (or rounding behavior) $\mathcal{R}_i \in \{5, 10, 25, 50, 100\}$. An individual of rounding type $\mathcal{R}_i = x$ rounds all of his answers to probabilistic questions about stock returns to the nearest multiple of $x\%$ at every point in time. Individuals' rounding types are independent of their (μ_i, σ_i) and I use $\vec{\wp} = \{\wp_5, \wp_{10}, \wp_{25}, \wp_{50}, \wp_{100}\}$ to denote the frequencies of rounding types across the population.

On every survey wave, a person might be asked to estimate the chances of positive returns, returns greater than 20%, or returns lower than -20%. In the model, person i's responses to these questions at time t are:

$$P_{i,t}^{\geq 0} = \left[\Phi\left(\frac{\mu_i}{\sigma_i} + \varepsilon_{i,t}^{\geq 0}\right)\right]_{\mathcal{R}_i},$$

$$P_{i,t}^{\geq 20} = \left[\Phi\left(\frac{\mu_i - \ln 1.20}{\sigma_i} + \varepsilon_{i,t}^{\geq 20}\right)\right]_{\mathcal{R}_i}, \tag{1}$$

$$P_{i,t}^{\leq -20} = \left[\Phi\left(\frac{\ln 0.8 - \mu_i}{\sigma_i} + \varepsilon_{i,t}^{\leq -20}\right)\right]_{\mathcal{R}_i},$$

where the operator $[\cdot]_x$ rounds its argument to the nearest multiple of x%, $\Phi(\cdot)$ is the univariate standard normal CDF, and the random disturbances $\left\{\varepsilon_{i,t}^{\geq 0}, \varepsilon_{i,t}^{\geq 20}, \varepsilon_{i,t}^{\leq -20}\right\}'$ follow the joint normal distribution $\mathcal{N}(\vec{0}, \Sigma)$. Without rounding or disturbances, Equation 1 would imply that people perfectly calculate and report the queried moments of their subjective log-normal distribution every wave—their answers would not change over time. The time-specific random disturbances represent survey errors and the effects of short-term information that might shift people's responses but not their long-term beliefs about stock-returns.

3.1.2 *The estimated distribution of beliefs*

I estimate the model using households' probabilistic assessments about stock returns in the HRS from 2002 to 2018. The estimated distributions of persistent beliefs about stock returns feature substantial heterogeneity and imply average beliefs that are pessimistic compared to the historical experience. The average person thinks that stock log-returns have a lower mean and a higher volatility than the S&P 500 index has had historically. These patterns are associated with educational attainment. The average college graduate has beliefs that imply greater and less volatile log-returns to stocks than the beliefs of his less-educated counterpart. People with higher levels of education are also more likely to respond more precisely to probabilistic questions, rounding their answers to finer degrees on average.

I represent the distribution of beliefs across the population Ω using discrete equiprobable grids of (μ, σ) pairs. The grids are equiprobable discretizations of bi-variate normal distributions that condition on the event that subjective standard deviations must be positive:

$$\begin{bmatrix}\mu_i \\ \sigma_i\end{bmatrix} \overset{\texttt{discretized}}{\sim} \mathcal{N}\left(\begin{bmatrix}\nu_\mu \\ \nu_\sigma\end{bmatrix}, \begin{bmatrix}\Psi_{1,1} & \Psi_{1,2} \\ \Psi_{2,1} & \Psi_{2,2}\end{bmatrix}\right) \mid \sigma_i > 0. \tag{2}$$

For any set of parameters $\{\nu_\mu, \nu_\sigma, \Psi\}$, I produce a set of 25 equiprobable (μ, σ) pairs that approximate the conditioned normal distribution in Equation 2; Appendix A discusses the discretization procedure in detail. There are two main advantages of this representation. First, it is flexible enough to accommodate distributions where beliefs have different averages, levels of dispersion, and correlations between subjective means and standard deviations. Second, I can incorporate the resulting discrete set of estimated (μ, σ) into a life-cycle model as a set of possible "belief types."

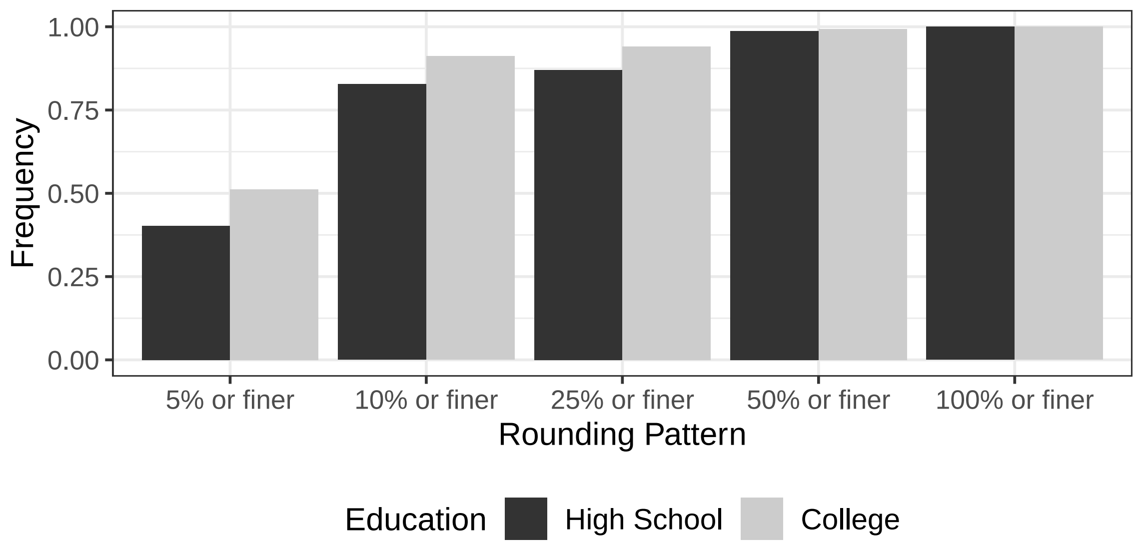
I estimate the model by maximum likelihood, using responses to the three probabilistic assessments in the nine biennial waves of the HRS between 2002 and 2018. The full set of parameters to estimate in my representation of beliefs comprises those in the distribution of μ and σ in Equation 2, the covariance matrix of random disturbances Σ, and the prevalence of different rounding types $\vec{\wp}$:

$$\vartheta^{\mathrm{B}} \equiv \{\nu_\mu, \nu_\sigma, \Psi, \Sigma, \vec{\wp}\}.$$

Appendix A shows how I derive the likelihood function from Equations 1 and 2; my derivation follows those of Kézdi and Willis (2011) and Ameriks, Kézdi, et al. (2020). My estimation sample consists of all person-year observations in which: a) the person being interviewed is the financial respondent of the household, b) the person being interviewed is between 50 and 65 years old, and c) the respondent did not refuse to answer nor answer "do not know" to any of the three questions regarding future stock returns.[18] These restrictions, in addition to considering only respondents with at least a high-school degree, yield a sample of $35,211$ individual-wave observations from $12,025$ unique individuals.

Because of the relationship between education and probabilistic assessments documented in Section 2.2, I estimate the model separately for people with different levels of educational attainment. I split my sample of high-school graduates into those with and without a college degree. Estimating the model separately lets people with different levels of educational attainment have different average beliefs, levels of disagreement, rounding patterns, and covariance structures in the random disturbances of their responses. Previous studies have found evidence of these differences (see, e.g., Kézdi and Willis 2011; Das, Kuhnen, and Nagel 2020; Ameriks, Kézdi, et al. 2020). A relationship between be-

[18]Out of the person-year observations that satisfy a) and b), the fraction of observations that I drop for not satisfying c) are 11.8% for those with a high-school degree and 5.2% for college graduates.



The bars in this figure are calculated using the estimated prevalence/frequencies of rounding types at every level of educational attainment. These point estimates can be found in Table A. Each bar is the sum of the frequencies of rounding types at least as fine as the one specified in the x axis. For example, "5% or finer" presents $\wp_5$, and "50% or finer" presents $\wp_5 + \wp_{10} + \wp_{25} + \wp_{50}$.

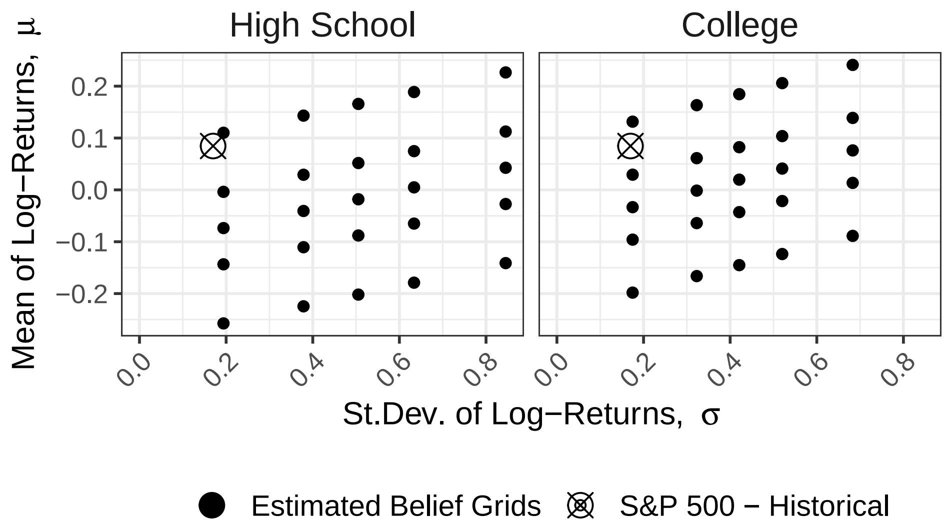
Figure 4: Estimated prevalence of rounding types

liefs about stock returns and education could explain part of the educational gradients in stockholding documented in Section 2.

Figure 4 presents the estimated frequencies of different rounding types for each level of educational attainment. The estimates suggest that different degrees of rounding are prevalent in the data and that, as found by Giustinelli, Manski, and Molinari (2022), more educated individuals tend to round their answers to finer levels. The fraction of individuals belonging to the finest rounding type (multiples of 5%) is 40% for high-school graduates and 51% for college graduates. Coarse rounding is non-negligible: 17% of high-school graduates and 9% of college graduates round their answers to levels coarser than 10% (25%, 50% or 100%).

The estimated models assign a large role to persistent differences in people's beliefs for explaining their probabilistic assessments. While the functional forms that I impose assume that (μ_i, σ_i) are individually-fixed, they do not limit the scale of their distribution across the population.[19] Figure 5 depicts the estimated grids of possible (μ, σ) pairs for every level of education, showing that they span large ranges of the mean-variance space. This large degree of variation suggests meaningful differences in the average responses

[19] In principle, the model could adjudicate the differences in responses to the time-specific shocks $\left\{\varepsilon_{i,t}^{\geq 0}, \varepsilon_{i,t}^{\geq 20}, \varepsilon_{i,t}^{\leq -20}\right\}'$ and find narrow distributions for (μ_i, σ_i).



The belief grids are equiprobable discretizations of the joint distribution in Equation 2, see Appendix A for details. The S&P 500 point depicts the average and standard deviation of annual log-returns to that index between 1881 and 2018. The data for this calculation comes from the accompanying data file to Chapter 26 of Shiller (1990).

Figure 5: Estimated belief grids

that different people give over time, which I interpret through my model as persistent differences in their subjective representations of stock returns. This finding relates to Giglio et al. (2021), who show that between 40% and 60% of the variation in the measured expectations of a sample of U.S. investors comes from individual "fixed-effects."

For every level of educational attainment, the estimated distributions of persistent beliefs imply that most people are more pessimistic about future stock returns than standard calibrations based on the historical experience. Figure 5 compares the estimated distribution of beliefs with the historical moments of log-returns to the S&P500 index. For both levels of educational attainment, the majority of points lie below and to the right of the S&P500. This means that most people believe log-returns to stock investments are lower on average and more volatile than those historically experienced by the S&P500. Moreover, many of the points fall below the $\mu = 0$ line, suggesting that a considerable fraction of the population believes that average log-returns are in fact negative. As suggested by Dominitz and Manski (2007), this "pessimism" about stock returns could help explain why a fraction of U.S. households do not own stocks at all, despite having substantial wealth (see Figure 2).[20]

[20]In other countries, studies have also shown that stock-market participation is insensitive to wealth windfalls (Andersen and Nielsen 2011; Briggs et al. 2021). These facts are difficult to accommodate for models that try to explain non-participation using monetary costs. For instance, Catherine (2021) assumes

Table 6: Summary statistics of estimated beliefs about stock returns

	$\mu_i = E_i[\ln \tilde{R}]$		$\sigma_i = \sqrt{V_i(\ln \tilde{R})}$		$E_i[\tilde{R}] - R$	Sharpe Ratio*	
	Mean	S.D.	Mean	S.D.	Fract > 0	Mean	S.D.
Estimated Beliefs							
High School	-0.017	0.131	0.511	0.221	0.60	0.281	0.110
College	0.020	0.118	0.424	0.172	0.72	0.265	0.169
Historical Realizations							
S&P 500 (1881-2018)	0.085	-	0.170	-	-	0.323	-

*The Sharpe ratio and its summary statistics are computed only for those beliefs for which it is positive. All summary statistics are taken over the points in the estimated beliefs grids for every level of education, depicted in Figure 5. The Sharpe ratios are computed with the nominal "risk-free" return factor as a benchmark. I take the average yearly risk-free return factor between 1881 and 2018 from the accompanying data file to Chapter 26 of Shiller (1990), which is 1.044.

The estimated distributions of beliefs imply a steep relationship between educational attainment and anticipated rewards from investing in stocks. Table 6 presents summary statistics of the distribution of persistent beliefs for every level of educational attainment. Expected log-returns (μ) vary widely within educational attainment groups, with within-group standard deviations of $1,300$ and $1,200$ basis points for high-school and college graduates, respectively. Average expected log-returns are higher for college graduates (200 basis points) than for high-school graduates (-170 basis points). Subjective assessments of volatility (σ) vary considerably within educational-attainment groups but also show a relationship with education: both their mean and standard deviation across individuals are lower for college graduates. The average subjective standard deviation of log-returns is $4,200$ basis points for college graduates and $5,100$ basis points for high-school graduates. Both values are much greater than the historical standard deviation of the S&P500's returns, which has been around $1,700$ basis points.

The estimated distributions of beliefs imply differences in people's expected rewards from the risks associated with stock-market participation. For high-school and college graduates, these differences align qualitatively with their differing investment patterns. The fourth column of Table 6 shows that not all individuals believe that the expected returns to stocks are greater than those of a safe bond: only 60% of high-school graduates and 72% of college graduates do. For those who expect a premium from stocks, the fifth and sixth column calculate the Sharpe ratio, which measures the expected excess returns

a fraction of the population—which he estimates to be 46%-47%—exogenously avoids the stock market. Pessimistic beliefs could be behind this persistent non-participation.

per unit of risk that they believe stocks offer. The average Sharpe ratios of high-school and college graduates who believe there is an equity premium are similar (0.28 and 0.26 respectively) but lower than the historical Sharpe ratio of the S&P500 index which—based on 1881 to 2018 data from Shiller (1990)—has been around 0.32.

Qualitatively, the fact that not all households believe that there is an equity premium could explain why some of them do not invest in stocks. The low perceived Sharpe ratios could explain why those who do own stocks invest a limited share of their wealth in them. In addition, the educational differences in beliefs about returns are also consistent with the different stockholding patterns of high-school and college graduates. Differences in the share of respondents who believe there is an equity premium could explain the relationship between education and participation rates. The similarity in the subjective Sharpe ratios of those who believe there is a premium could explain why the conditional shares of wealth in stocks are similar for high-school and college graduates. To evaluate these possibilities quantitatively, I now present a life-cycle model of saving and portfolio choices.

3.2 Life-Cycle Model of Saving and Portfolio Choices

The life-cycle model has several features in common with portfolio choice models in the literature (see, e.g., Cocco, Gomes, and Maenhout 2005; Gomes and Michaelides 2005; Campanale, Fugazza, and Gomes 2015; Fagereng, Gottlieb, and Guiso 2017; Catherine 2021). Households save to smooth their consumption against fluctuations in their income, which come from deterministic changes as they age and random shocks, both permanent and transitory. My model features a bequest motive and age-varying health-expenditure shocks as additional reasons for saving; both are important motives in explaining post-retirement wealth (De Nardi, French, and Jones 2010; Ameriks, Briggs, et al. 2020). Agents face two different financial frictions when deciding how to allocate their savings between assets. First, they must pay a monetary cost before owning stocks for the first time. Second, they face a 10% early-withdrawal penalty when liquidating stocks before retirement.

I now discuss the main components of the model and leave its full mathematical description and treatment for Appendix C.

3.2.1 *Lifespan, utility, and mortality*

Time periods in the model represent a year. Agents enter the model at age 24 and can live up to a maximum age of 100. At the end of every year, they face an exogenous risk of death that becomes certain at the maximum age. The probability of surviving from age

t to $t+1$ is represented by δ_{t+1}, while the probability of not surviving is represented by $\not\delta_{t+1} \equiv 1 - \delta_{t+1}$.

Agents derive utility from consumption. Their utility function follows a constant relative risk-aversion specification: a level of consumption C gives the agent instant utility

$$u(C) = \frac{C^{1-\rho}}{1-\rho}, \tag{3}$$

where ρ is the coefficient of relative risk-aversion.

If an agent dies at the end of a year, after making all his choices, he derives warm-glow utility from bequeathing his total wealth. The utility derived from bequeathing wealth x is:

$$\mathbb{B}(x) = \mathbb{b} \times \frac{(x/\mathbb{b})^{1-\rho}}{1-\rho} = \mathbb{b}^{\rho} \times \frac{x^{1-\rho}}{1-\rho} = \mathbb{b}^{\rho} \times u(x),$$

where $\mathbb{b} \geq 0$ is a parameter that controls the intensity of the bequest motive. This is the same specification used by, e.g., Gomes and Michaelides (2005).

3.2.2 *Income process*

Agents supply labor inelastically and retire exogenously at the end of the year in which they turn 65. Their labor earnings, denoted by $Y_{i,t}$, are a product of two factors: a permanent component represented by $P_{i,t}$ and a transitory stochastic component represented by $\theta_{i,t}$. Labor earnings and their permanent component follow:

$$\begin{aligned} \ln Y_{i,t} &= \ln P_{i,t} + \ln \theta_{i,t} \\ \ln P_{i,t} &= \ln P_{i,t-1} + \ln \Gamma_{i,t} + \ln \psi_{i,t}, \end{aligned}$$

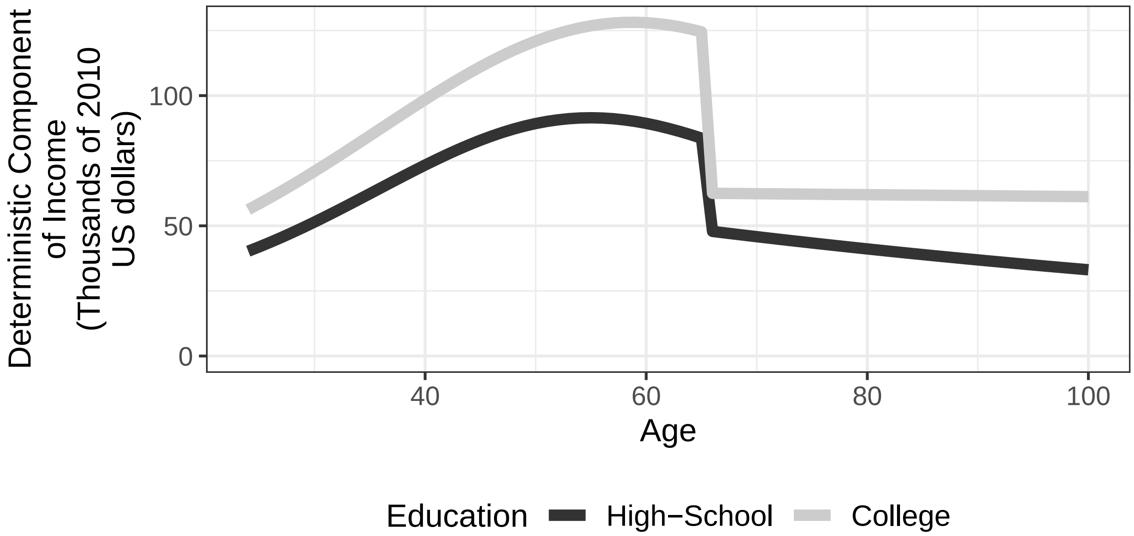
where Γ_t is a deterministic growth factor that captures life-cycle patterns in earnings, and $\ln \psi_{i,t} \sim \mathcal{N}(-\sigma^2_{\psi}/2, \sigma^2_{\psi})$ is a multiplicative shock to permanent income.[21]

The transitory component of earnings $\theta_{i,t}$ is a mixture that represents unemployment and temporal fluctuations in income that occur while employed:

$$\ln \theta_{i,t} = \begin{cases} \ln \mathcal{U}, & \text{With probability } \mho \\ \ln \tilde{\theta}_{i,t} \sim \mathcal{N}(-\sigma^2_{\theta}/2, \sigma^2_{\theta}), & \text{With probability } 1 - \mho. \end{cases}$$

$\mho$ denotes the probability of unemployment, and $\mathcal{U}$ denotes the replacement factor of unemployment benefits.

[21]The mean of the shock is set so that $E[\psi_{i,t}] = 1$.



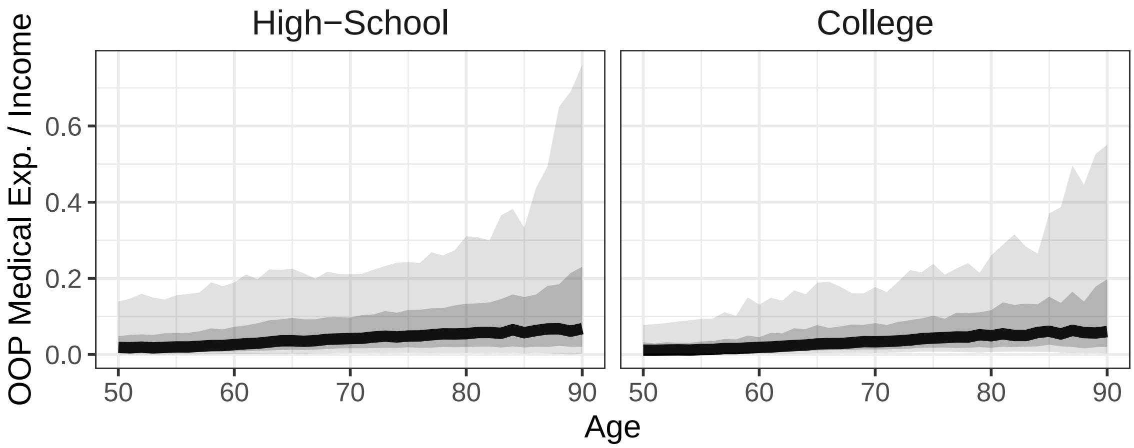
These estimated trajectories for the deterministic component of permanent income come from Cagetti (2003). I thank the author for sharing his exact estimates.

Figure 6: Deterministic component of income by education level

The sequences of growth factors $\{\Gamma_t\}_{t=25}^{100}$ differ dependent on the individual's highest level of education. I take their values from Cagetti (2003). Figure 6 displays the income paths that individuals with different education levels would experience in the absence of shocks.[22] The decline after age 65 corresponds to retirement. I also take the volatilities of transitory and permanent income shocks (σ_ψ and σ_θ) used by Cagetti (2003), which come from Carroll and Samwick's (1997) estimates.

After retirement, individuals are no longer subject to transitory and permanent shocks to their earnings. Instead, they face out-of-pocket medical-expenditure shocks. As people age, medical expenditures increase rapidly (see Figure 7). The anticipation of these rising expenditures has been shown to be one of the reasons why the elderly do not spend their wealth as quickly as a basic life-cycle model would predict (De Nardi, French, and Jones 2010; Ameriks, Briggs, et al. 2020). The literature that specializes in the study of these expenditures has identified several important features, such as their dependence on persistent health states (Kopecky and Koreshkova 2014; Ameriks, Briggs, et al. 2020), their relationship with permanent income (De Nardi, French, and Jones 2010), and the prevalence of "catastrophic" shocks (French and Jones 2004). To incorporate these shocks into my model, I adopt a parsimonious representation that matches the distribution of expenditures across the population and matches the fact that they increase with age and

[22]This is $P_{i,24} \times \prod_{j=25}^{t} \Gamma_{i,t}$.



The figure depicts estimates of the distribution of people's out-of-pocket medical expenditures expressed as a ratio of their annual income. At every age, the solid line represents the median, the dark shaded area represents the 25th and 75th percentiles, and the light-shaded area represents the 10th and 90th percentiles.

Figure 7: Out-of-pocket medical expenditures over the life cycle

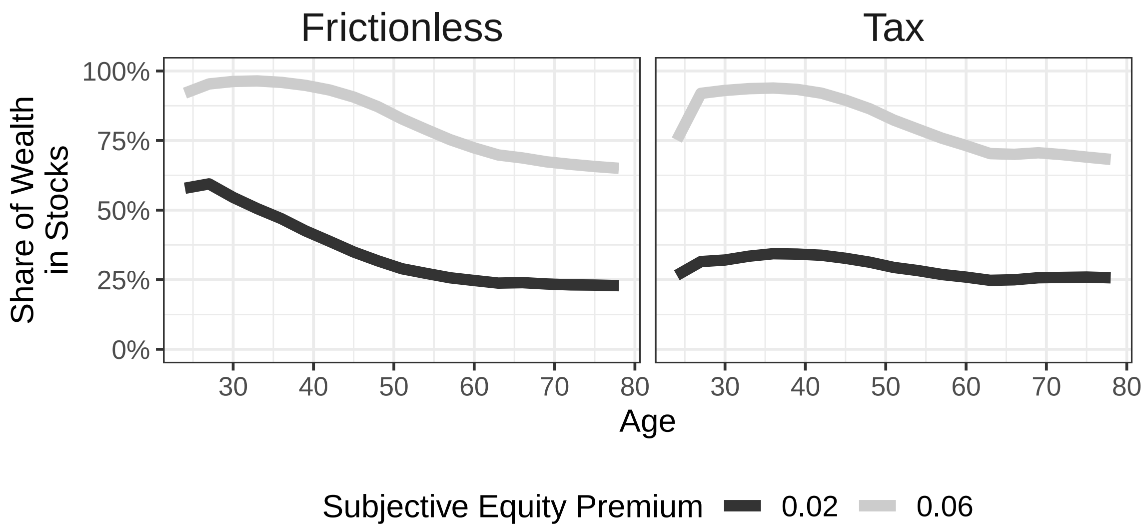
permanent income. Every year, agents draw a shock $\texttt{oop}_{i,t}$ that represents the fraction of their earnings used up by out-of-pocket medical expenses. I assume that government programs cover any health expense above an agent's income, so that income net of medical expenses cannot be negative. The process for earnings net of costs and permanent income becomes:

$$
\begin{aligned}
Y_{i,t} &= P_{i,t} \times \max\{0.0, 1 - \texttt{oop}_{i,t}\} \\
P_{i,t} &= P_{i,t-1}\Gamma_i.
\end{aligned}
$$

The shocks $\texttt{oop}_{i,t}$ are independent across time and follow age- and education-specific distributions that approximate the patterns in Figure 7. I calibrate these distributions using the RAND HRS longitudinal file; I describe the process in Appendix B.

3.2.3 *Financial assets and frictions*

Agents try to smooth their consumption by saving, and they have two assets available for this purpose. The first asset is a risk-free liquid account with a constant per-period return factor R. The second asset is a stock-fund with a stochastic return factor $\tilde{R}$ that agents view as log-normally distributed and independent across time. I denote the dollar amounts available to agent i at the start of period t in the risk-free account and the stock-fund with $M_{i,t}$ and $N_{i,t}$, respectively. The flows between the two assets are one of the agents' control variables and denoted with $D_{i,t}$, with $D_{i,t} > 0$, representing a movement of funds from



Lines in the figure correspond to the average share of wealth in stocks conditional on participation for agents simulated under two different parametrizations of the model. For the simulations, I use $\rho = 5$ and $\beta = 0.96$ and the income process of those with a high-school degree. "Frictionless" corresponds to a version of the model in which there are no re-balancing taxes ($\tau = 0$). "Tax" is a version in which $\tau = 0.1$.

Figure 8: Financial frictions and their interaction with beliefs.

the risk-free to the risky account.

The model has four different financial frictions. First, agents cannot short-sell any of the assets or borrow against their future income. Second, agents enter the model not having access to the stock-fund and must pay a one-time financial cost to access it. As in Gomes and Michaelides (2005), the cost represents the money and time spent opening a brokerage account and getting familiarized with the stock-market. The cost is proportional to agents' permanent income, $P_{i,t} \times F$, where F is a parameter to be estimated. Third, withdrawals from the stock-fund are taxed at a constant rate $\tau = 0.1$ before agents retire.[23] This friction represents early retirement-fund withdrawal penalties and the costs associated with liquidating stock positions. Finally, agents must pay for their consumption using funds from their risk-free accounts only.

As demonstrated by Campanale, Fugazza, and Gomes (2015), the combination of rebalancing penalties and the fact that consumption must be paid for using risk-free funds generates a reason for young people to not allocate all of their wealth to stocks, as is predicted by standard life-cycle portfolio models (see Cocco, Gomes, and Maenhout 2005). An agent who anticipates the possibility of consuming part of his savings in the next period—because of an unemployment spell, for instance—might keep a buffer of

[23] The withdrawal tax rate becomes $\tau = 0$ after agents retire.

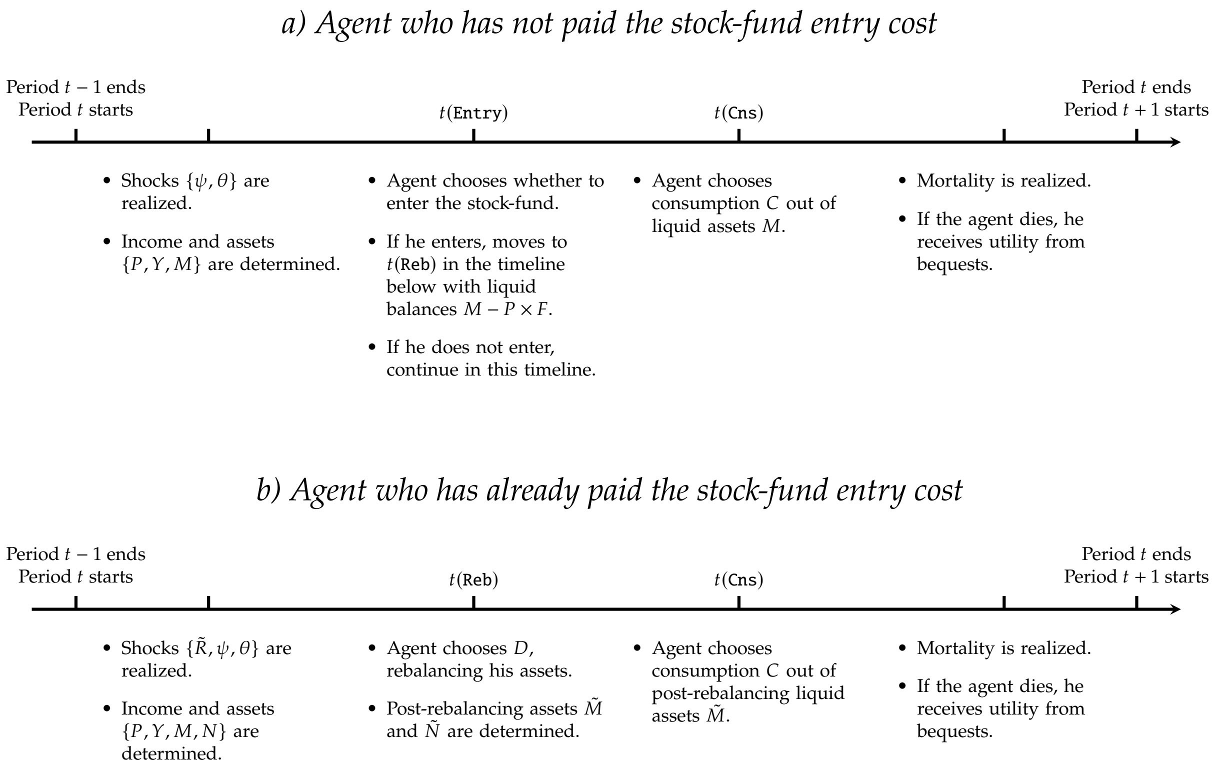


Figure 9: Summary of timing in the life cycle model

risk-free funds to avoid having to pay the stock-withdrawal tax if this is the case. The size of the desired buffer will depend, among other factors, on the agent's beliefs about the equity premium, the volatility of his income, and the magnitude of the withdrawal tax. Figure 8 uses simulations from the model under different parametrizations to illustrate this fact. The figure shows that, by themselves, pessimistic beliefs about the equity premium produce a downward shift in the share of wealth in stocks of stock-market participants. In spite of the downward shift, the share of wealth in stocks preserves its steeply declining profile over the life cycle, which is not a feature of U.S. household portfolios (see Figure 1). Introducing the withdrawal tax has only a minor effect on the share of wealth in stocks when agents are optimistic about the premium—a one-time penalty of 10% is small when weighted against a 6% yearly equity premium. In contrast, the tax substantially lowers the share of wealth in stocks for young agents who believe the equity premium is 2%, resulting in a flatter profile over the life cycle that resembles U.S. data more closely.

3.2.4 *Timing and recursive representation*

Figure 9 summarizes the timing of stochastic shocks and optimizing decisions that occur within a period of the life cycle-model. Agents enter the model in timeline a), not having paid the stock-fund entry cost $P_{i,t} \times F$. They are presented with the option to pay the cost and enter the fund every year. Once they pay the cost, they move to the portfolio-rebalancing stage $t(\texttt{Reb})$ of timeline b) and remain on timeline b) for the rest of their lives.

To illustrate the choices and constraints faced by agents succinctly, Equation 4 presents the recursive-form value function of an agent who has paid the financial participation cost and therefore has access to the stock-fund.[24] I present the value function of the agent who has not paid the financial participation cost in Appendix C, which also discusses various alternative representations of the model that I use in its solution.

$$
\begin{aligned}
V_t^{\texttt{In}}(M_t, N_t, P_t) = \max_{C_t, D_t} u(C_t) + \beta\delta_{t+1}\mathbb{E}_t\left[V_{t+1}^{\texttt{In}}(M_{t+1}, N_{t+1}, P_{t+1})\right] \\
+ \not{\delta}_{t+1}\mathbb{B}(A_t + \tilde{N}_t) \\
\text{Subject to:} \\
-N_t \leq D_t \leq M_t, \quad 0 \leq C_t \leq \tilde{M}_t \\
\tilde{M}_t = & M_t - D_t\left(1 - 1_{[D_t \leq 0]}\tau\right) \\
\tilde{N}_t = & N_t + D_t \\
A_t = & \tilde{M}_t - C_t \\
M_{t+1} = & RA_t + Y_{t+1} \\
N_{t+1} = & \tilde{R}_{t+1}\tilde{N}_t \\
P_{t+1} = & \Gamma_{t+1}\psi_{t+1}P_t \\
Y_{t+1} = & \theta_{t+1}P_{t+1}
\end{aligned}
\quad . \tag{4}
$$

4. Estimation

To determine whether survey-measurements of beliefs can improve the life-cycle model's capacity to fit U.S. households' savings and portfolios, I now estimate the model under alternative specifications of beliefs about stock returns. In the first specification, all the

[24]Individual subindices i are dropped for simplicity.

simulated agents believe that future stock-fund returns will follow a distribution that approximates their historical behavior—this is the commonly used *full-information rational-expectations* (F.I.R.E.) specification. In the second specification, the agents' beliefs about future stock-returns are heterogeneous and distributed across the population following the specifications that I estimated from survey measurements in Section 3.1.2. For each level of education and each specification of beliefs, I estimate the model's unobservable parameters that govern agents' preferences and barriers to stock-market participation. The estimation strategy searches for the parameters that best replicate the life-cycle profiles of U.S. households' savings, stock-market participation rates, and shares of financial wealth in stocks.

4.1 Data, Sample Restrictions, and Targeted Variables

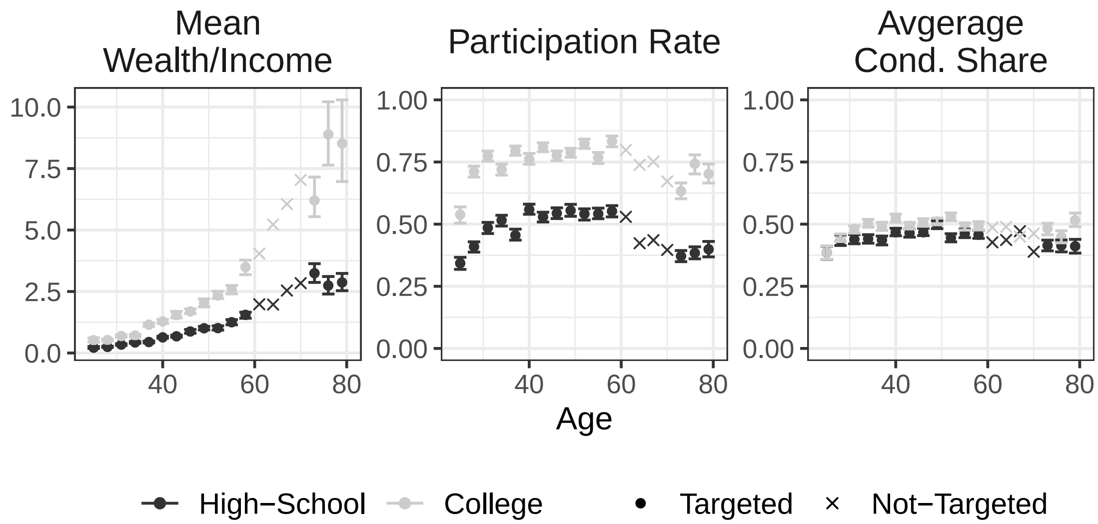
The estimation exercise targets the life-cycle profiles of wealth-to-income ratios, stock-market participation rates, and the shares of wealth in stocks conditional on participation of U.S. households. I construct these targeted variables using the nine waves of the *Survey of Consumer Finances* (SCF) between 1995 and 2019. My variables of interest rely on definitions and calculations in the SCF's *summary files*, which produce standard measures of households' wealth and income from the raw survey.[25]

The *wealth-to-income ratio* measures the financial savings of an individual relative to what would be their "usual" income—their permanent income in the life-cycle model. For my measure of wealth, I take the economic unit's total financial assets (`fin`). For my measure of income, I take the sum of wage and salary income (`wageinc`) and social security and pension income (`ssretinc`).

To measure stock holdings, I use the SCF's estimated total value of financial assets invested in stocks (`equity`), which includes direct and indirect investments. I define *stock market participation* as a binary variable that takes the value of one if the given economic unit has stock holdings greater than zero, and zero otherwise. Finally, I calculate the *share of wealth in stocks* as a unit's stock holdings divided by their wealth; if a unit's wealth is zero, I set the share to zero.

In order to generate a sample that more closely matches the type of household represented in the model and to ensure that the variables of interest are well-defined, I apply various filters to the data before using it to compute the targeted moments. First, I keep only economic units whose respondent was born between 1920 and 1995. To make the wealth-to-income ratio comparable to a model-analogue that uses permanent income, I

[25] Variables in `teletype font` denote calculations that are readily available in the summary files.



Each point represents the relevant statistic computed over the group of SCF respondents with the given level of educational attainment and falling in a 3-year age bin. See the main text for definitions of the sample and relevant variables. Error bands correspond to 95% point-wise confidence intervals calculated using 500 bootstrapped samples for each level of education.

Figure 10: Targeted moments

only keep units who report that their income was "normal" in the given year.[26] To ensure that the ratio is defined, I only keep units with positive incomes. Finally, I exclude business owners from the sample.[27]

I group observations by the respondent's level of education and age. For education, I split the sample into those without a high-school degree (which I do not analyze), those with a high-school but not a college degree, and those with a college degree. For age, I form three-year bins starting at age 24 and up to a maximum age of 80, for a total of 19 groups: $\{24, 25, 26\}$, $\{27, 28, 90\}$, ..., $\{75, 76, 77\}$, $\{78, 79, 80\}$. The moments that my estimation routine targets are summary statistics of the variables of interest calculated over the education-by-age groups.

For each group of observations, the moments that I target in estimation are:

- **Average wealth-to-income ratio**: the average of the wealth-to-income ratio. The ratio can take on extreme values for agents with a low measured wage income. To limit the influence of these extreme observations, I winsorize the wealth-to-income ratios at their within-group 95th percentile before taking their average.

[26]The question of whether income was unusual was added in the 1995 wave. For this reason, I exclude previous waves from this part of the analysis.

[27]I define business owners as those with bus > 0.

Table 7: Non-estimated parameter values

Symbol	Interpretation	Value	Source
$\mathbb{b}$	Bequest Intensity	2.5	Gomes and Michaelides (2005)
$\{\delta_t\}_{t=0}^{T}$	Survival probabilities	-	S.S.A. Actuarial tables 2010[28]
$\{\Gamma_t\}_{t=0}^{T}$	Permanent income drift (educ.)	-	Cagetti (2003)
σ_ψ, σ_θ	Volatility of income shocks (educ.)	-	Carroll and Samwick (1997)
$\mho$	Probability of unemployment	0.050	-
$\mathcal{U}$	Unemp. benefits replacement factor	0.500	National median, Ganong, Noel, and Vavra (2020)
π	Log-inflation rate	0.024	Mean CPI Log-Inflation.
r	Log risk-free rate (nominal)	0.043	Mean 1-year U.S. bond log-returns.
R	Risk-free return factor (real)	1.019	$\exp\{r-\pi\}$
μ^{SP500}	Mean stock log-return	0.085	S&P500 Index (nominal).
σ^{SP500}	St. Dev. stock log-returns	0.170	S&P500 Index (nominal).

Parameters that depend on educational attainment are marked with "(educ.)." Averages and standard deviations of financial variables are all taken over the 1881-2018 period. The data on financial assets and the CPI index come from the 'Chapter 26' file in Robert Shiller's website: http://www.econ.yale.edu/ shiller/data.htm.

- **Stock-market participation rate**: the average of the binary stock-market-participation variable.
- **Average conditional share of wealth in stocks**: the average share of wealth in stocks of those who participate in the stock market.

I use survey weights for the calculation of these moments and re-scale the weights of different waves so that each of them has an equal representation in the moments' calculations.

Figure 10 displays each of the moments for every education-by-age group. The figure also presents 95% confidence intervals for each targeted moment. I determine the confidence intervals by calculating the targeted moments on 500 bootstrapped samples for each level of educational attainment. The error bars correspond to the 2.5th and 97.5th percentiles of the bootstrapped values of each moment. The bars demonstrate that the only moments with considerable sampling variation are the post-retirement wealth ratios of college graduates.

4.2 Objective Function and Optimization

For every level of education and specification of beliefs, I estimate the preferences and participation costs that minimize the distance between the targeted moments in the SCF and their model-implied counterparts.

The set of parameters that I estimate structurally consists of the coefficient of risk-aversion (ρ), the time-discount factor (β), and the size of the cost of accessing the stock-fund for the first time (F). I denote this set of parameters with $\vartheta \equiv \{\rho, \beta, F\}$. I set other parameters related to the income process and mortality to historical estimates or values from the literature, and they remain fixed throughout the estimation process; I summarize their values or sources in Table 7. The parameters that govern the returns to different assets and agents' expectations about them are discussed in detail in the next section.

For any given set of parameters, I solve the life-cycle model and simulate populations of agents that I use to find model-implied counterparts to the targeted moments. I solve the model by backward induction using a combination of the techniques outlined in Carroll (2006), Iskhakov et al. (2017), and Druedahl (2021); I describe the process in detail in Appendix E. I use the resulting policy functions to simulate populations of agents on which I calculate model-counterparts to the targeted empirical moments. The model is not well suited to accommodate the transitional dynamics of households' savings and portfolios as they move into retirement; it assumes that all agents retire exogenously at age 65. Therefore, I exclude the bins spanning ages 60 to 71, leaving a total of 15 targeted age-bins and 45 moments for each level of educational attainment.[29] For a level of educational attainment e, I use m_0^e to denote a vector of the 45 targeted empirical moments and $\hat{m}^e(\vartheta)$ to denote its model-implied counterpart under parameters ϑ.

The loss function that I minimize is:

$$L^e(\vartheta) = \left(m_0^e - \hat{m}^e(\vartheta)\right)' \mathbb{W}^e \left(m_0^e - \hat{m}^e(\vartheta)\right), \tag{5}$$

where $\mathbb{W}^e$ is a diagonal weighting matrix. Since the moments have different scales, I set $\mathbb{W}^e$ so that moment deviations are expressed as fractions of the average relevant statistic across age groups.[30] I obtain estimates as:

$$\hat{\vartheta}^e = \arg\min_{\vartheta} L^e(\vartheta). \tag{6}$$

To solve the minimization problem, I use the `TikTak` algorithm (Arnoud, Guvenen, and Kleineberg 2019) as implemented in the `estimagic` toolbox (Gabler 2022). I use 2,500 initial "exploration points" and allow for 10 full local-optimization runs using the `DFO-LS`

[29] 15 age-bins times three moments of interest (median wealth-to-income ratio, participation rate, and average conditional share of wealth in stocks).

[30] For example, for individuals with a college degree, the diagonal positions of $\mathbb{W}^{\text{Coll.}}$ that multiply errors in the stock market participation rate are set to $1/(\overline{\text{Part}})^2$ where $\overline{\text{Part}}$ is the average of the 15 stock market participation rates in $m_0^{\text{Coll.}}$.

algorithm (Cartis et al. 2019), which takes advantage of the least-squares structure of the optimization problem.

4.3 Agents' Expectations and Returns to Financial Assets

In the model, returns of the stock-fund follow a distribution that approximates the historical behavior of the S&P500 index. In the first specification of agents' beliefs, everyone's expectations are consistent with this data-generating process. In the second specification, agents' expectations match survey measurements instead.

The simulated log-returns of the stock-fund follow a normal distribution. Their mean and variance match those of the S&P500's log-returns between 1881 and 2018, and I make a constant adjustment for inflation π, which I take to be its average over the same period:

$$\ln \tilde{R} \sim \mathcal{N}(\mu^{SP500} - \pi, \sigma^{SP500}). \tag{7}$$

Table 7 presents the values of μ^{SP500}, σ^{SP500}, and π.

The first specification of beliefs that I use is *full-information rational-expectations* (F.I.R.E.). Under this specification, agents have correct beliefs about the stock-fund's returns (Equation 7) when solving their dynamic optimization problem. For each level of education, I simulate populations of 500 agents that run for 1,000 years and use them to compute the model's counterparts to targeted moments, $\hat{m}^e(\vartheta)$. The second specification of beliefs that I use is *estimated beliefs*. Under this specification, I replace every agent of the F.I.R.E. simulation with 25 agents whose beliefs about the stock-fund's returns come from the distributions I estimated in Section 3.1.2.[31] The jth agent believes that $\ln \tilde{R} \sim \mathcal{N}(\hat{\mu}_j - \pi, \hat{\sigma}_j)$, where $\{\hat{\mu}_j, \hat{\sigma}_j\}_{j=1}^{25}$ is the grid of estimated beliefs for the given level of educational attainment, displayed in Figure 5. I use the resulting populations of 12,500 agents to compute model-implied moments.

5. Life-Cycle Cycle Model Estimates

The results from structurally estimating the life-cycle model confirm that incorporating survey measurements of beliefs improves the model's capacity to explain the savings and portfolios of U.S. households. For both high-school and college graduates, replacing model-consistent (F.I.R.E.) beliefs with a specification that fits survey measurements reduces the distance between the model's predictions and the targeted moments of the data.

[31] The 25 estimated-beliefs agents share the same shock realizations of the F.I.R.E. agent they are replacing. They differ only in their beliefs about stock-fund returns.

Table 8: S.M.M. estimated parameters under different belief models

	College		High-School	
	F.I.R.E.	Est. Beliefs	F.I.R.E.	Est. Beliefs
CRRA (ρ)	11.396	5.114	8.607	4.231
	[11.373; 11.529]	[5.059; 5.125]	[8.541; 8.647]	[4.192; 4.254]
Disc. Fact. (β)	0.634	0.886	0.331	0.761
	[0.624; 0.643]	[0.882; 0.892]	[0.321; 0.345]	[0.752; 0.771]
Entry Cost ($F \times 100$)	1.041	0.000	3.116	2.576
	[0.308; 1.698]	[0.000; 0.000]	[3.015; 3.400]	[2.310; 2.706]
SMM Loss, $L^e(\hat{\vartheta})$	5.264	2.857	15.984	3.998

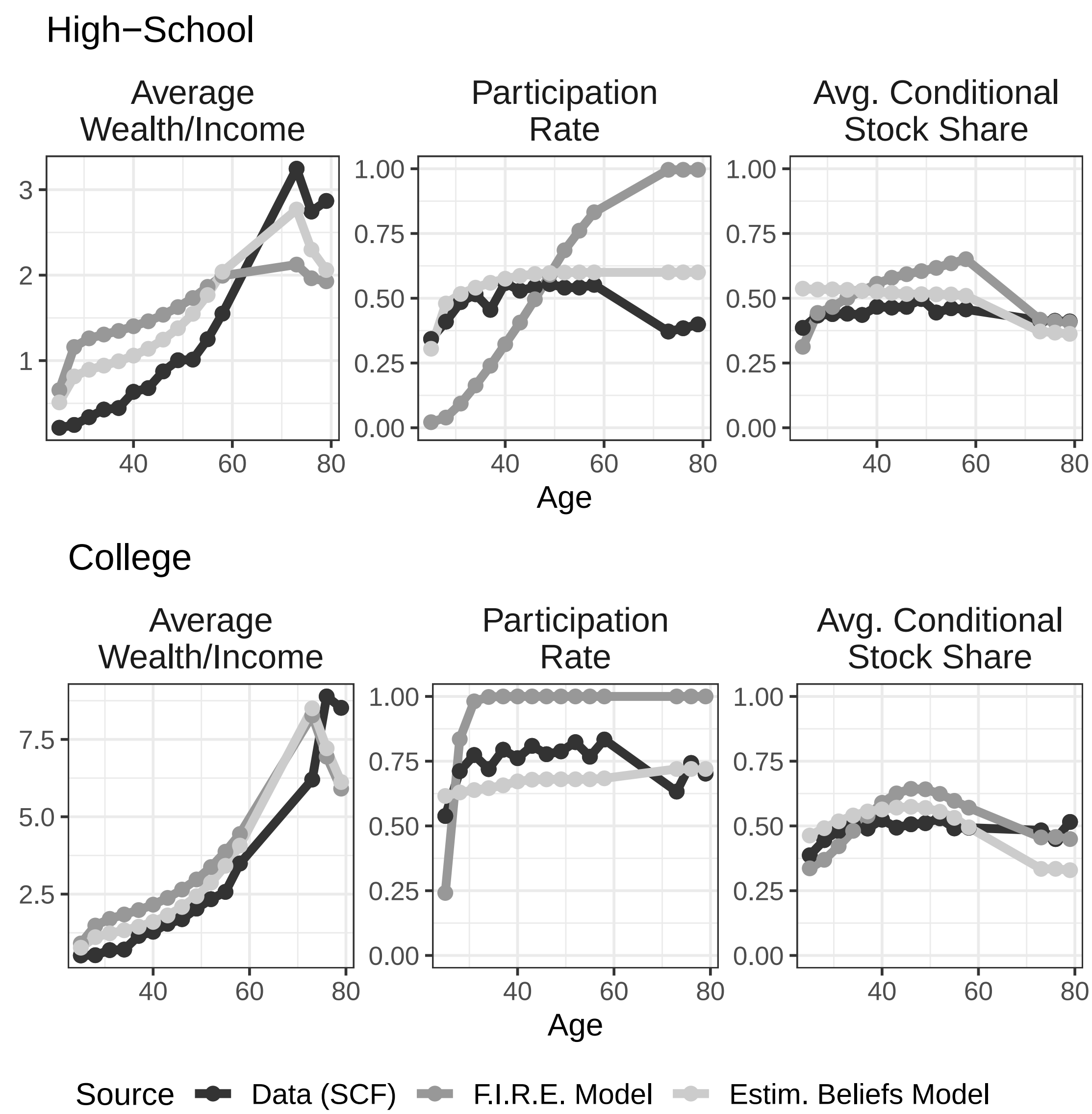
The brackets under each point estimate are 95% confidence intervals that come from estimating a surrogate model on bootstrapped moments, see Appendix F for details. "F.I.R.E" stands for full-information rational-expectations and "Est. Beliefs" corresponds to the heterogeneous beliefs specification, both described in Section 4. The "SMM Loss" row displays the value of the Simulated Method of Moments loss function (Equation 5) attained by the given parameter values and belief specifications.

In both cases, the improvement comes from the capacity to fit low participation rates with low participation costs and low portfolio shares with moderate levels of risk-aversion. This is possible because in the estimated distribution of beliefs not everyone thinks that there is an equity premium, and those who do underestimate the risk-return compensation that the stock market offers.

For both levels of educational attainment that I consider, the life cycle model fits the targeted moments more closely when it uses estimated beliefs instead of F.I.R.E. beliefs. Table 8 displays the estimated parameters for each level of educational attainment and specification of beliefs, along with the loss function (Equation 6) evaluated at the estimates. The loss function aggregates squared differences between the model-implied and empirical moments; thus, it serves as an index of how well each model fits the age-profiles of wealth-to-income ratios, stock-market participation rates, and conditional shares of wealth in stocks. Specifications that use the beliefs estimated from survey measurements have lower losses than their F.I.R.E. counterparts. The reductions are substantial: 75% for high-school graduates (15.984 to 3.998) and 46% for college graduates (5.264 to 2.857).

Figure 11 compares the predicted moments of different model specifications with their empirical counterparts, revealing the sources of the improvement in their measured fit.

The F.I.R.E. model does not replicate the humped-shaped participation rates of high-school graduates, which peak at less than 60%. Instead, it produces participation rates that start at 2% and increase with age until they reach 100%. The reason is that, under this



Each dot in the figure represents the relevant statistic calculated over a three-year age-group. See the main text for precise definitions of the age groups and statistics. Ages 60 to 71 are omitted because of transitional dynamics into retirement that the model does not account for.

Figure 11: Life-cycle model's fit of targeted moments

specification of beliefs, every agent who saves even modest amounts wants to participate, and the only way to prevent them from doing so is to impose high entry costs. The costs can only generate participation rates that increase with age because, once an agent has paid them, he can participate for the rest of his life. Therefore, to generate a participation profile with a low peak of less than 60%, the F.I.R.E. model would need high entry costs that would make participation at younger ages counterfactually low. Instead, the best-fitting entry cost of 3.1% of annual income generates participation rates that do not match the shape of the true age-profile but whose average across age-bins (51%) is close to that in the data (48%).

In contrast, the model that uses the estimated beliefs specification accurately reproduces the participation rates of high-school graduates before retirement. Because not all agents believe that there is an equity premium under this specification, the model generates participation rates that are moderate—this would be the case even in the absence of entry costs. The model fits increasing participation rates between ages 24 and 40 using a cost of 2.6% of permanent income. After age 40, the model uses the fraction of the population who does not believe in an equity premium to match the plateauing of participation rates until age 60. Without these agents, participation would continue to grow. Neither specification of beliefs can replicate the decline in participation rates after retirement that occurs in the data. This is due to the structure of costs: since there are no per-period costs associated with owning stocks, agents who already participate have little incentive to completely exit the stock market.

For college graduates, neither specification of beliefs replicates stock market participation rates perfectly: the F.I.R.E. model overestimates them, and the estimated beliefs model underestimates them. As was the case with high-school graduates, the F.I.R.E. model is constrained by the fact that all agents with sufficient savings want to own stocks. For college graduates, it uses a cost of 1% of annual permanent income that reduces early participation, but all agents overcome this cost by age 40, leading to a 100% participation rate for most of the life cycle. The opposite problem occurs in the estimated beliefs model: the fraction of agents that think there is an equity premium is lower than the actual participation rates of college graduates aged 40 to 60. Therefore, despite its null estimated participation cost, this model underestimates participation rates for most of the life-cycle.

The differences between the wealth-to-income ratios and conditional stock shares generated by the two specifications of beliefs are subtler. For high-school graduates, both models overestimate the wealth-to-income ratios of young agents and underestimate those of retirees; these errors are greater for the F.I.R.E. model. Similar issues appear in the wealth-to-income ratio of college graduates but to a lesser degree, with both specifications of beliefs tracking the empirical age-profile more closely. The average conditional shares of wealth in stocks produced by the models are close to the empirical age profiles for both levels of educational attainment and both specifications of beliefs. The profiles are flatter than those predicted by baseline frictionless calibrations, (see Figure 8) and this brings them closer to the empirical profiles. The main noticeable discrepancy in the model with estimated beliefs is the reduction of the average conditional share of wealth in stocks of college graduates after retirement. This reduction comes from a change in the composition of participants when the rebalancing tax is removed: some agents with pessimistic beliefs (but who still think that there is an equity premium) enter the market, and they drive down the average conditional share.

While the two specifications of beliefs produce qualitatively similar wealth-to-income ratios and conditional shares of wealth in stocks, they rely on different mechanisms to generate them. The main difference is how the two specifications reduce the conditional share of wealth in stocks to the moderate levels observed in the data. Models with F.I.R.E. beliefs rely on high relative risk aversion coefficients of $\rho = 8.6$ for high-school graduates and $\rho = 11.4$ for college graduates. Since all agents believe in a large equity premium under this specification, the only way to dissuade participants from allocating large shares of their savings to stocks is to make them extremely risk-averse. On the other hand, under the estimated beliefs specification, Table 6 shows how even agents who believe in an equity premium think (on average) that the risk-return trade-off offered by stocks is not as attractive as historical benchmarks suggest. This feature enables the specification with estimated beliefs to match the empirical age-profiles of the conditional shares of wealth in stocks with lower relative risk aversion coefficients of $\rho = 4.2$ for high-school graduates and $\rho = 5.1$ for college graduates.

The different relative risk aversion coefficients required by the F.I.R.E. and estimated-beliefs specifications produce differences in how the models fit wealth-to-income ratios. High relative risk aversion coefficients increase agents' precautionary saving, preventing the models from producing agents with low wealth. This is evident in Figure 11, where both belief specifications struggle to match the savings of younger agents, especially high-school graduates. To counteract the effect of precautionary saving on the wealth of young agents, the models use lower time-discount factors (β) than those typically found in the macroeconomics and labor-economics literature.[32] This effect is stronger for F.I.R.E.

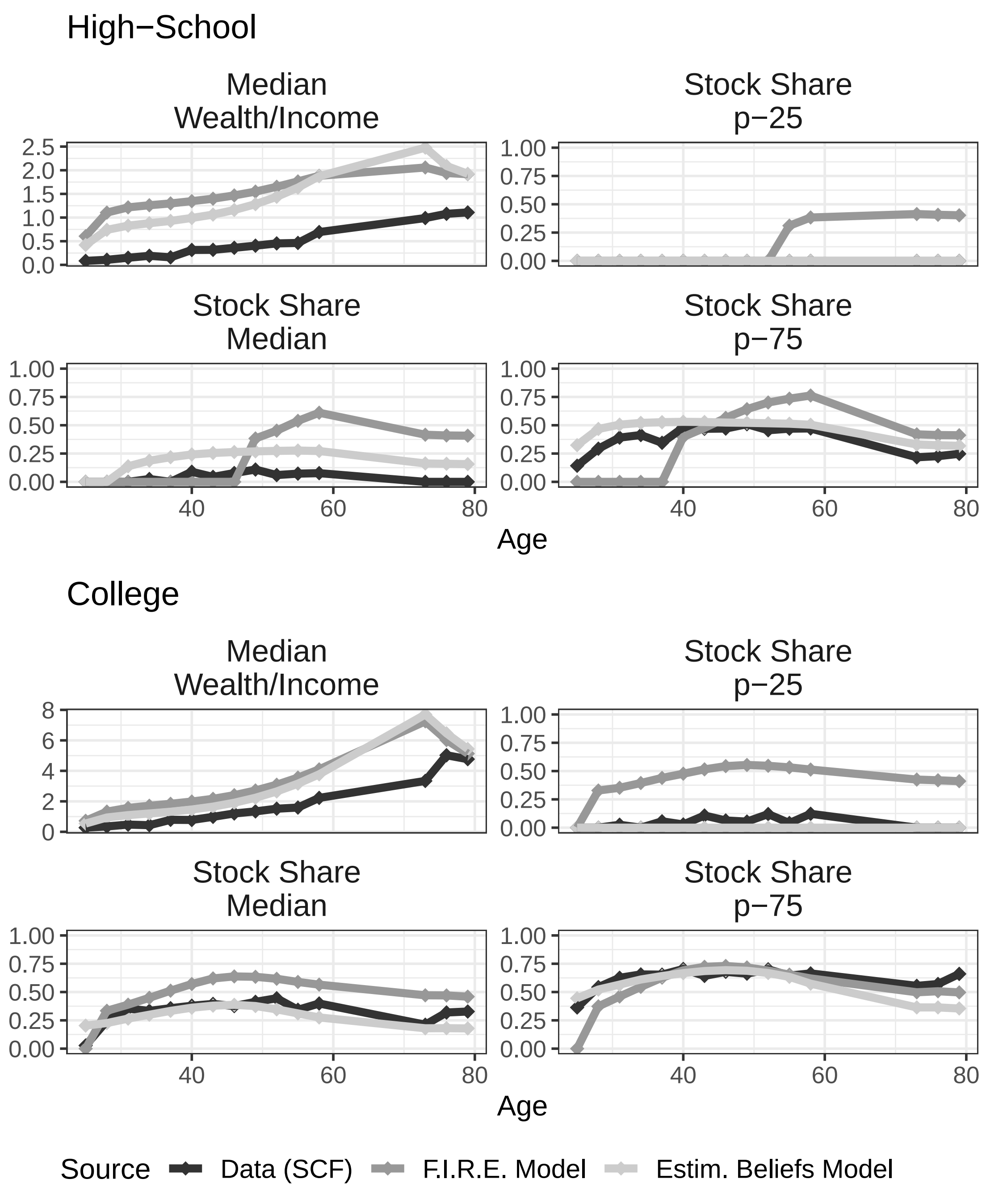
[32]Most studies that use consumption-saving models to match wealth find annual discount factors $\beta > 0.9$, see e.g., Gourinchas and Parker (2002), Cagetti (2003), De Nardi, French, and Jones (2010), and Carroll, Slacalek, et al. (2017). However, because of the higher relative risk aversion coefficients needed to match portfolio shares, lower estimated discount factors are frequently found in the household finance literature, see e.g., Fagereng, Gottlieb, and Guiso (2017) and Catherine (2021).

models because of their higher relative risk aversion coefficients; it results in $\beta = 0.33$ for high-school graduates and $\beta = 0.63$ for college graduates, implying that agents discount their future utility at rates of 67% and 37% per year respectively. The models that use the estimated beliefs instead can afford higher time-discount factors—$\beta = 0.76$ for high-school graduates and $\beta = 0.89$ for college graduates—because of the lower pressure of conditional portfolio shares on the relative risk aversion coefficients.

In sum, replacing F.I.R.E. beliefs with the estimated specification of beliefs produces a better fit of targeted moments, lower estimates of relative-risk aversion coefficients, higher estimates of discount factors, and lower estimates of entry costs. These conclusions are robust to the sampling variation of the targeted moments shown in Figure 10. In Appendix F, I follow Chen, Didisheim, and Scheidegger 2021 and Catherine et al. 2022 in approximating the structural life-cycle models with deep neural networks. I use the approximate models to demonstrate that these qualitative conclusions would hold if the estimation exercise was repeated for 500 different bootstrapped values of the targeted parameters. The confidence intervals in Table 8 come from the distributions of these bootstrapped estimates.

Figure 12 presents the models' predictions for features of the data not directly targeted in estimation. The age profiles of median wealth-to-income ratios reveal that the models underestimate the skewness of savings for all specifications of beliefs and levels of educational attainment—they roughly match their means but overestimate their lower medians. The figure also shows different percentiles of the unconditional share of wealth in stocks, which the estimated beliefs specification models fit better than their F.I.R.E. counterparts. The F.I.R.E. models struggle to produce agents that participate in the stock market but invest a low share of their wealth in stocks. Because they rely on the participation cost, the age profiles that they imply for unconditional shares tend to start at 0%—before the agent pays the cost—and then jump to higher values when agents decide to enter. Additionally, they do not generate much variation in the share of wealth in stocks of those who participate; the different percentiles of the unconditional share are close at most ages. In contrast, the models with estimated beliefs generate a distribution for the unconditional share whose different percentiles follow qualitatively different trajectories across the life cycle—the levels and shapes of the 25th and 75th percentiles are different. The different percentiles implied by the models with estimated beliefs track their empirical counterparts closely for both high-school and college graduates.

Overall, using the specification of beliefs estimated using survey measurements improves the life-cycle model's fit of U.S. households' portfolios with moderate levels of risk-aversion, lower financial costs of entry, and higher time-discount factors than its



Each marker in the figure represents the relevant statistic calculated over a three-year age-group. See the main text for precise definitions of the age groups. None of the moments in this figure were targeted in estimation. Ages 60 to 71 are omitted because of transitional dynamics into retirement that the model does not account for.

Figure 12: Life-cycle model's fit of non-targeted moments

F.I.R.E. counterparts. The main challenge that remains, common to both belief specification strategies, is how to explain the prevalence of households with low savings. This fact is difficult to reconcile, considering the levels of risk aversion needed to match the portfolio allocations of such households. This difficulty arises because, in the model, agents rely only on their savings to insure against consumption fluctuations. Therefore, if high risk aversion makes these individuals inclined to avoid fluctuations in their portfolios, this same risk aversion also encourages them to save. Various features that the model omits could help resolve this difficulty; some that the literature has explored are social assistance programs, access to debt, heterogeneous preferences, and housing.

6. The Welfare Costs of Misspecified Beliefs

This section considers a scenario in which stocks continue to perform as they have historically. In this scenario, the beliefs estimated from survey measurements would be misspecified, because they differ from the historical distribution of returns. An agent with misspecified beliefs about returns has a lower expected level of welfare than an identical counterpart with accurate beliefs, as the former's decisions are based on an incorrect model of the world. I apply the estimated life-cycle model to quantify these welfare shortfalls for agents with the beliefs estimated from survey measurements. The metric that I use quantifies welfare shortfalls as the fraction of permanent income that agents would give up in exchange for correctly specified beliefs.[33] Average welfare shortfalls start out at less than 2% of permanent income for young agents but follow a "hump" shape that peaks before retirement at 3.60% to 7.20% of permanent income, depending on their level of education. The shortfalls vary across agents' levels of education and wealth, as well as their respective beliefs about stock returns.

Because the agents' beliefs can differ from the true data-generating process for risky returns, the discounted welfare that *they* expect can differ from the discounted welfare that an *objective observer*—one who knew the true data-generating processes and the agents' decision rules—would expect them to receive. Let:

$$\mathfrak{V}_t(P_t, M_t, N_t, \texttt{Paid}_t; \mu, \sigma)$$

denote the discounted welfare that we objectively expect an agent to receive starting from

[33]This calculation is performed from the point of view of an objective planner; agents are not aware that their beliefs are misspecified.

age t and state $(P_t, M_t, N_t, \texttt{Paid}_t)$, if his beliefs about returns are (μ, σ).[34] The function $\mathfrak{V}_t(\cdot)$, which I define formally in Appendix G, uses the agent's preferences to discount the future and assumes that he will follow the policy functions that solve his dynamic problem according to his beliefs. However, $\mathfrak{V}_t(\cdot)$ uses the true data-generating parameters $(\mu^{SP500}, \sigma^{SP500})$ in its expectations, and thus, if $(\mu, \sigma) \neq (\mu^{SP500}, \sigma^{SP500})$, then $\mathfrak{V}_t(\cdot; \mu, \sigma)$ will not correspond to the agent's value function.

To measure the expected welfare shortfalls that an agent suffers from his misspecified beliefs, I use a compensating variation in terms of his permanent income. The metric corresponds to the proportional reduction in permanent income (present and future) that would bring the agent to his current expected welfare, if his beliefs were corrected. Formally, for an agent with state variables $(P_t, M_t, N_t, \texttt{Paid}_t)$ and beliefs (μ, σ), I find the λ that satisfies:

$$\mathfrak{V}_t\left(P_t, M_t, N_t, \texttt{Paid}_t; \mu, \sigma\right) = \mathfrak{V}_t\left((1-\lambda) \times P_t, M_t, N_t, \texttt{Paid}_t; \mu^{SP500}, \sigma^{SP500}\right). \tag{8}$$

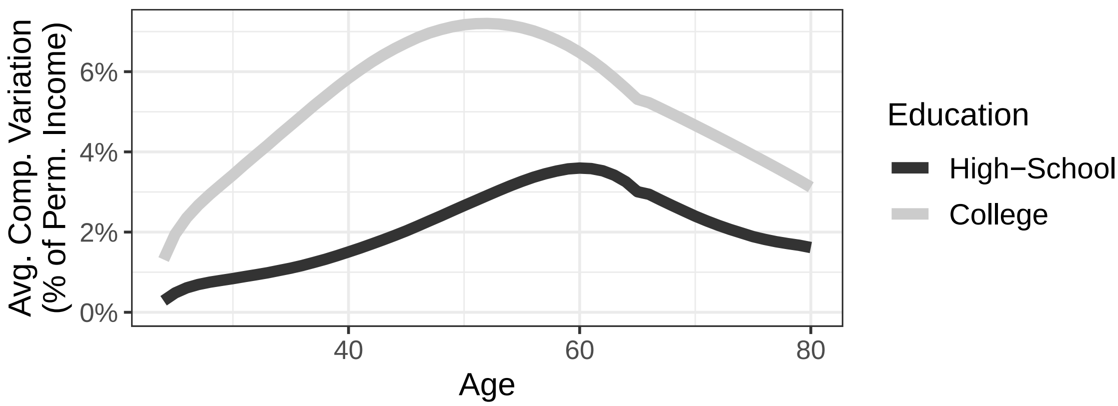
The metric λ can also be interpreted as the maximum fraction of permanent income that an altruistic and objective planner would be willing to take from the agent in exchange for amending his beliefs about returns. Equation 8 cannot be solved for λ analytically, but it shows that an agent's expected welfare shortfall is contingent on his age, permanent income, assets, whether or not he has paid the entry-cost, and his beliefs.[35]

I calculate the expected welfare shortfalls from the estimated heterogeneous beliefs for individuals with the preferences and assets implied by the estimated life-cycle model. For each level of educational attainment, I find the expected-welfare functions $\{\mathfrak{V}_t(\cdot)\}_{t=24}^{100}$ using the preference and cost parameters from Table 8 for the estimated beliefs specifications. Then, I simulate the lives of agents with beliefs about stock returns drawn from the education-specific estimated distributions. These simulations use the same population sizes and shock realizations as in the estimation process (see Section 4). Finally, I evaluate the estimated welfare shortfall λ of every agent at every year.

The welfare shortfalls from misspecified beliefs follow a hump shape across the life cycle, starting low in agents' youth and peaking between the ages of 52 and 60, prior to retirement. Figure 13 presents the expected welfare shortfalls λ of high-school and college graduates at every age, evaluated at their simulated assets and incomes. For both levels of education, the average shortfall at age 24 is below 2% of their permanent income. The reason is that the benefits from accurate beliefs are reaped later in life, when agents have

[34] $\texttt{Paid}_t$ indicates whether the agent has paid the fixed stock-market entry cost or not.

[35] Because the model is homothetic in permanent income, λ can be expressed as a function of assets normalized by permanent income, and the other states.

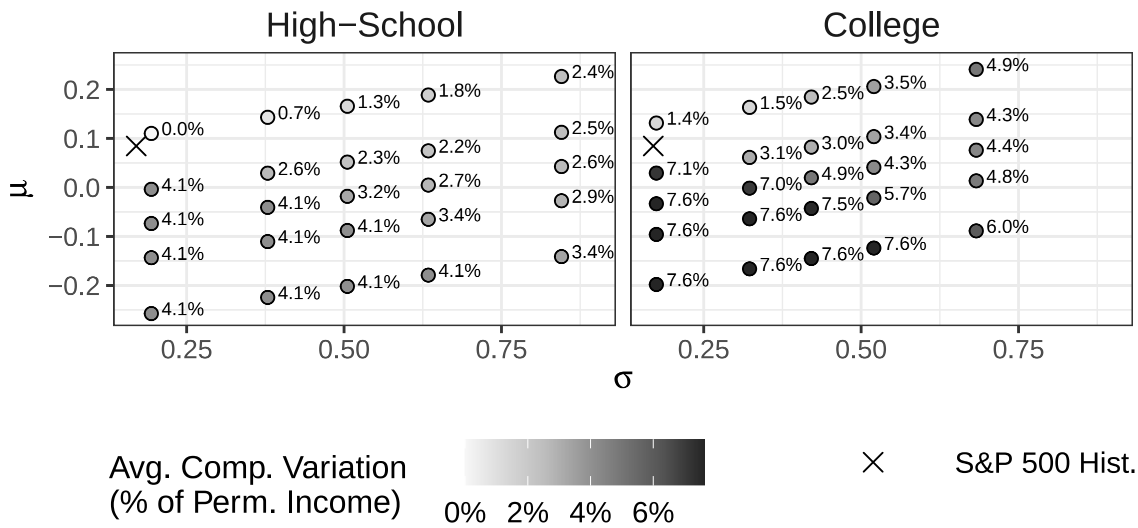


The figure presents average welfare losses from misspecified beliefs about risky-asset returns at every age. I simulate populations of agents that behave according to the beliefs and preferences estimated in Sections 3.1.2 and 5. Then, for every agent-period observation, I compute the expected welfare shortfall λ defined in Equation 8. I report the average of this measure for every age-education combination.

Figure 13: Expected welfare shortfalls across the life cycle

accumulated more savings and they rely on them for their consumption. The impatient discounting of these future benefits results in the low initial welfare shortfalls. However, as agents accumulate wealth and approach retirement—when their savings become a more important source of their consumption—the shortfalls progressively increase, reaching their peak at age 52 for college graduates with a value of 7.20% of permanent income and at age 60 for high-school graduates with a value of 3.60% of permanent income. Thereafter, the shortfalls progressively decline as agents deplete their wealth and their life expectancy declines. Despite college graduates having beliefs closer on average to the historical benchmark (see Table 6), their average welfare shortfalls are greater than those of high-school graduates at all ages. The greater shortfalls of college graduates are due to their higher estimated discount factor, higher levels of savings, and lower replacement rates of retirement income.

The age patterns and educational differences in the welfare shortfalls from distorted beliefs are consistent with findings from the financial literacy literature. First, Figure 13 indicates that agents would derive the greatest (discounted) benefits from correcting their misconceptions about the risky asset in the period between age 50 and retirement. Empirically, different measures of financial literacy follow a similar hump shape across the life cycle of U.S. respondents and peak in this age-range (Lusardi and Mitchell 2023). Additionally, in models that allow for endogenous accumulation of financial knowledge, investments in financial knowledge peak at this ages and are greater for college graduates than high-school graduates (Lusardi, Michaud, and Mitchell 2017, 2020). These patterns



The figure presents average welfare losses from misspecified beliefs about risky-asset returns for every set of beliefs considered in the life-cycle model estimation. I simulate populations of agents that behave according to the beliefs and preferences estimated in Sections 3.1.2 and 5. Then, for every agent-period observation, I compute the expected welfare shortfall λ defined in Equation 8. I report the average of this measure for every belief-education combination at age 65.

Figure 14: Expected welfare shortfalls from different beliefs at age 65

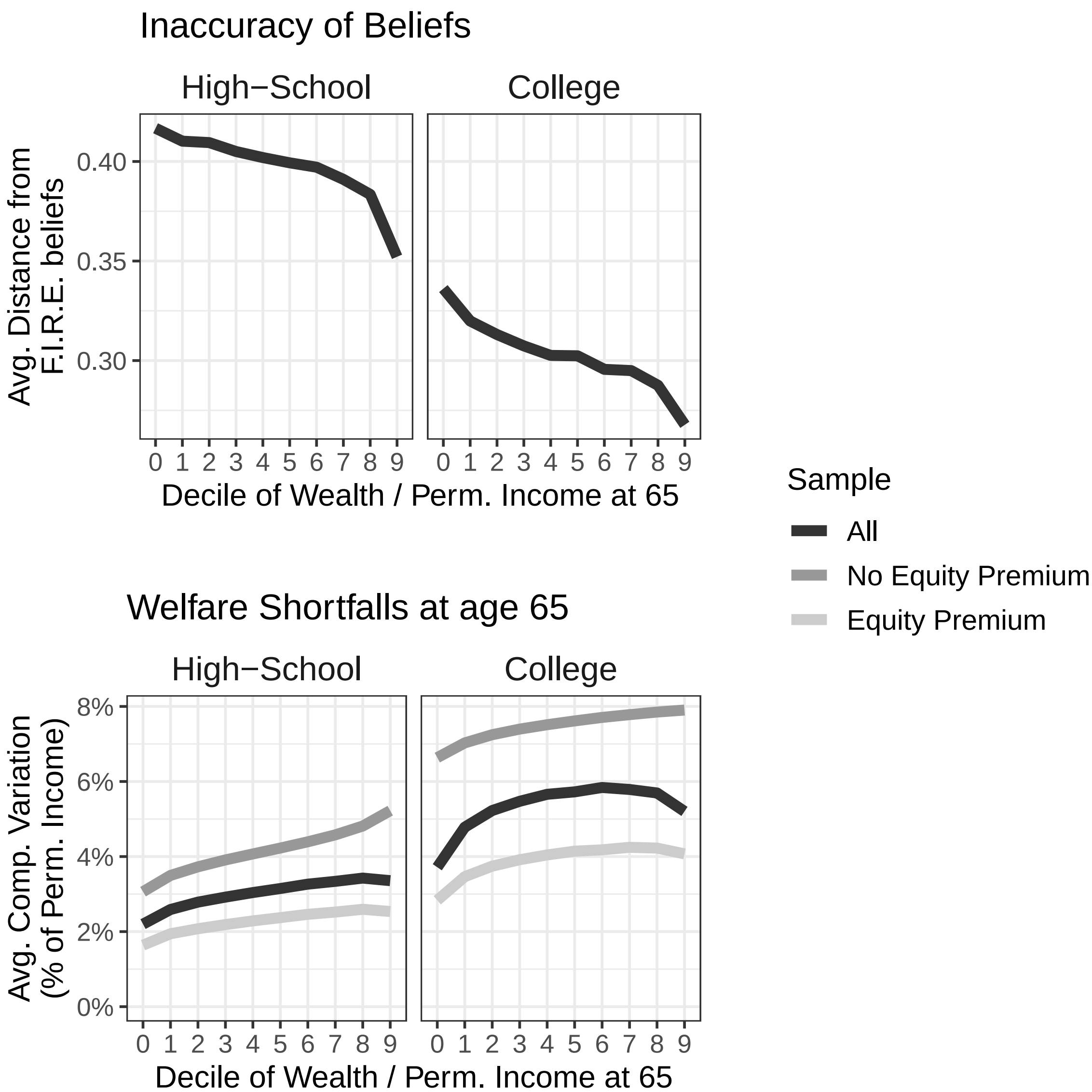
highlight the years preceding retirement as a potential "teachable moment" for financial knowledge interventions since people have accumulated enough wealth to put their knowledge to use and anticipate that they will become more reliant on this knowledge for their support. Identifying these "teachable moments" has been found to be a crucial determinant of the success of these interventions in changing downstream behaviors (Kaiser and Menkhoff 2017). Workplace interventions, for instance, are a modality of financial knowledge program that has gathered increasing interest (see Clark 2023; Lusardi and Mitchell 2023) and which these results favor over earlier interventions.

The shortfalls in welfare also vary substantially with individuals' beliefs about stock returns, and the largest impacts fall on those with beliefs that discourage them from ever owning stocks. Figure 14 displays the average expected welfare shortfall for simulated individuals with different beliefs and levels of education at age 65, their last working year. For both levels of education, the largest welfare shortfalls occur among individuals with low subjective means and volatilities of log-returns. These individuals' beliefs imply that they do not perceive an equity premium; therefore, they refrain from ever participating in the stock market. The average welfare shortfall for these groups of non-participants is 4.12% of permanent income for high-school graduates and 7.63% for college graduates. Welfare shortfalls are lower for the group of agents who perceive an equity premium, progressively increasing as beliefs move further away from the model-consistent benchmark. The average welfare shortfall among those who believe there is an equity premium is 2.27% of permanent income for high-school graduates and 4.41% for college graduates.

While wealth increases an agent's potential benefits from having accurate beliefs, it is precisely agents with more accurate beliefs who accumulate more wealth in the first place; the combination of these two forces results in a muted cross-sectional relationship between wealth and welfare shortfalls from inaccurate beliefs. The top row of Figure 15 shows that agents who reach age 65 with greater wealth have beliefs about stock returns (μ, σ) closer to model-consistent F.I.R.E. beliefs $(\mu^{SP500}, \sigma^{SP500})$ on average. The reason is that more accurate beliefs make agents invest greater shares of their wealth in stocks throughout their lives and, having benefited from greater returns, they reach the age of retirement with higher wealth.[36] The bottom row of Figure 15 presents the average welfare shortfall λ of agents across the wealth distribution, distinguishing between those whose beliefs imply that there is or there is not an equity premium. The unconditional relationship between welfare losses and wealth is muted: the average losses in the lowest and highest wealth deciles are 2.20% and 3.36% of permanent income for high-school graduates and 3.72% and 5.20% for college graduates. The fact that the average individual has more accurate beliefs in higher wealth deciles plays a role in attenuating the relationship. Instead, examining the group of agents who do not think that there is an equity premium reveals substantial losses that are greater for wealthier agents. The average loss for this sub-group in the highest wealth decile reaches 5.22% of permanent income for high-school graduates and 7.90% for college graduates.

The results in this section highlight considerations for the design and application of interventions aimed at changing financial knowledge and behaviors. Lusardi and Mitchell (2023) stress the importance of pinpointing the groups for which these interventions have the greatest effects and suggest that they might not be cost-effective for groups that will not use their financial knowledge in a timely way. The foregoing analysis suggests that interventions aimed at improving the financial knowledge of young and disadvantaged individuals with low savings may generate only modest differences in their welfare, even if they succeed at enhancing their knowledge. This conclusion is consistent with the findings of, e.g., Lusardi, Michaud, and Mitchell (2020). Targeting interventions to the wealthiest individuals can also deliver muted impacts, because they tend to already have

[36] More accurate beliefs could, in principle, make agents want to lower their exposure to stocks. However, in the estimated distributions of beliefs, most agents underestimate the mean and overestimate the volatility of log-returns. Thus, beliefs that are closer to model-consistent generally imply that stocks are more attractive.



The distance from F.I.R.E. beliefs is calculated for each individual as $\sqrt{(\mu - \mu^{SP500})^2 + (\sigma - \sigma^{SP500})^2}$. The welfare shortfall is the compensating variation λ defined in Equation 8. I simulate populations of agents that behave according to the beliefs and preferences estimated in Sections 3.1.2 and 5. I split them by their educational attainment and, at age 65, I rank them into deciles of their ratio of wealth to permanent income. I report the average of each measure for every education by wealth-bin combination. "All" uses all the agents of a given education-decile bin, "No Equity Premium" uses only those whose beliefs do not imply that there is an equity premium, and "Equity Premium" uses only those whose beliefs imply that there is an equity premium.

Figure 15: Belief distortions and welfare shortfalls across the wealth distribution

higher financial knowledge. Instead, this exercise suggests that the greatest impacts would come from improving the financial knowledge of 50-plus-year-old workers with substantial savings who do not participate in the stock market. The question of how to design interventions effective at changing both financial knowledge and behaviors remains a crucial area of ongoing research (see Kaiser and Menkhoff 2017; Kaiser, Lusardi, et al. 2022; Clark 2023).

7. Concluding Remarks

Dominitz and Manski (2007) note that "many households (...) are not as convinced as economists are about the existence of an equity premium." Since the their pioneering work, the HRS has expanded the range of available measurements of household expectations in both time and variety, having collected nearly two decades of measurements, and now including three different questions regarding equity returns. This paper exploits this expanded battery of measurements to show that many households remain unconvinced of the existence of an equity premium and that even those who seem to believe in its existence deem it smaller in risk-adjusted terms than what economists usually assume.

These facts about measured expectations provide a qualitative explanation for why most households do not invest most of their wealth in equities. The exercises carried out in this paper quantitatively evaluate the plausibility of this explanation. They demonstrate that the explanation has several attractive features: it substantially enhances the capacity of the considered model to reproduce both portfolio choices and savings and their relationship with age and education; it brings the estimates of unobserved preference parameters to ranges more consistent with alternative sources of evidence; and it is backed by a robust body of measurements and empirical results.

Important challenges and questions remain. The model put forward in this paper faces difficulties in matching the low savings of groups like young households and those without a college degree. Allowing households to borrow and modeling the social programs that low-wealth households use to smooth their consumption are possible ways to reduce these difficulties. Additionally, the model proposed in this paper assumes that households do not change their beliefs about equity returns or that they do not react to short-term fluctuations in their opinions. This assumption is made in order to replicate features of belief measurements like the "dominance of individual fixed effects" (Giglio et al. 2021) and the fact that households' portfolios have weak responses to changes in their elicited expectations. Questions such as why households do not learn in ways that eliminate the persistent heterogeneity in their measured beliefs, or why the association between changes in their elicited expectations and changes in their portfolios is weak, are left unresolved. The increasing availability of individual-level measurements of expectations and portfolios can help address these questions.

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Appendix

A. Estimating the Model of Beliefs

As discussed in Section 2.2 and in previous studies like Giustinelli, Manski, and Molinari (2022), people round their answers to probabilistic questions. For each of the probabilistic questions about stock returns, Table 3 shows the fraction of all answers that are multiples of 5%, 10%, 25%, 50% and 100%. The Table shows that, for each question, less than 2.5% of the answers are not multiples of 5%. Based on this fact, 5% is the finest level of rounding in my model, and I round the few answers that are not multiples of 5% to the nearest 5% multiple.

A.1 The Likelihood Function

Denote the set of parameters of the beliefs model with

$$\vartheta^{\mathrm{B}} \equiv \{\nu_\mu, \nu_\sigma, \Psi, \Sigma, \vec{\wp}\},$$

and let the data consist of triplets of responses to probabilistic questions

$$\left\{\left\{P^{\geq 0}_{i,t}, P^{\geq 20}_{i,t}, P^{\leq -20}_{i,t}\right\}_{t\in\mathcal{T}(i)}\right\}_{i\in\mathcal{I}}$$

where $\mathcal{I}$ is the set of respondents and $\mathcal{T}(i)$ denotes the set of time periods in which individual i answered the probabilistic questions.

The first step in evaluating the likelihood function for ϑ^{B} is to find the equiprobable grid for (μ, σ) that is associated with $\vartheta^{\mathrm{B.\,Grid}} \equiv \{\nu_\mu, \nu_\sigma, \Psi\}$. I construct an equiprobable grid with n^2 points that approximates distribution 2 using the following steps.

1. Find an equiprobable n-point grid for $x \sim \mathcal{N}(\nu_\sigma, \Psi_{2,2})|x > 0$. Denote the grid with $\sigma^{\#}$.

2. For every σ in $\sigma^{\#}$, find an equiprobable n-point grid for the distribution of μ_i conditional on $\sigma_i = \sigma$, which given Equation 2 is

$$\mathcal{N}\left(\nu_\mu + \frac{\Psi_{1,2}}{\Psi_{2,2}}(\sigma - \nu_\sigma)\,,\ \Psi_{1,1} - \frac{\Psi_{1,2}^2}{\Psi_{2,2}}\right).$$

   Denote that grid with $\mu^{\#}(\sigma)$.

3. The joint n^2-point grid will be

$$(\mu, \sigma)^{\#} \equiv \{(\mu, \sigma) : \mu \in \mu^{\#}(\sigma), \sigma \in \sigma^{\#}\}.$$

The likelihood of an agent's response depends both on the parameters of their subjective distribution of returns (μ_i, σ_i) and on the degree to which they round their answers. I denote agent i's level of rounding with $\mathcal{R}_i$ and consider 5%, 10%, 25%, 50%, and 100% as the possible levels to which agents round their answers. Therefore, $\forall i \;\; \mathcal{R}_i \in \{5, 10, 25, 50, 100\}$.

The likelihood of an answer that has been rounded is that of the interval of all the real numbers that round to that answer. To facilitate the representation of these intervals, define the following two sets of functions:

- $\underline{u}_{\mathcal{R}}(x)$ gives the lowest number in $[0, 1]$ that rounds to x when the level of rounding is $\mathcal{R}$. For instance, $\underline{u}_5(0.15) = 0.125$, $\underline{u}_{10}(0.30) = 0.25$, $\underline{u}_{25}(0.5) = 0.375$, $\underline{u}_{50}(100) = 0.75$, and $\underline{u}_{100}(0.0) = 0.0$.

- $\bar{u}_{\mathcal{R}}(x)$ gives the highest number in $[0, 1]$ that rounds to x when the level of rounding is $\mathcal{R}$. For instance, $\bar{u}_5(0.15) = 0.175$, $\bar{u}_{10}(0.30) = 0.35$, $\bar{u}_{25}(0.5) = 0.625$, $\bar{u}_{50}(100) = 1.0$, and $\bar{u}_{100}(0.0) = 0.5$.

With these functions and the response model from Equation 1, we can say that if agent i rounds their answers to the $\mathcal{R}_i$-level and has subjective-distribution parameters (μ_i, σ_i), then

$$\begin{aligned} P^{\geq 0}_{i,t} = x \leftrightarrow & \underline{u}_{\mathcal{R}_i}(x) \leq \Phi\left(\frac{\mu_i}{\sigma_i} + \varepsilon^{\geq 0}_{i,t}\right) \leq \bar{u}_{\mathcal{R}_i}(x) \\ \leftrightarrow & \Phi^{-1}(\underline{u}_{\mathcal{R}_i}(x)) - \frac{\mu_i}{\sigma_i} \leq \varepsilon^{\geq 0}_{i,t} \leq \Phi^{-1}(\bar{u}_{\mathcal{R}_i}(x)) - \frac{\mu_i}{\sigma_i}, \end{aligned} \tag{9}$$

$$\begin{aligned} P^{\geq 20}_{i,t} = y \leftrightarrow & \underline{u}_{\mathcal{R}_i}(y) \leq \Phi\left(\frac{\mu_i - \ln 1.20}{\sigma_i} + \varepsilon^{\geq 20}_{i,t}\right) \leq \bar{u}_{\mathcal{R}_i}(y) \\ \leftrightarrow & \Phi^{-1}(\underline{u}_{\mathcal{R}_i}(y)) - \frac{\mu_i - \ln 1.20}{\sigma_i} \leq \varepsilon^{\geq 20}_{i,t} \leq \Phi^{-1}(\bar{u}_{\mathcal{R}_i}(y)) - \frac{\mu_i - \ln 1.20}{\sigma_i}, \end{aligned} \tag{10}$$

and

$$\begin{aligned} P^{\leq -20}_{i,t} = z \leftrightarrow & \underline{u}_{\mathcal{R}_i}(z) \leq \Phi\left(\frac{\ln 0.8 - \mu_i}{\sigma_i} + \varepsilon^{\leq -20}_{i,t}\right) \leq \bar{u}_{\mathcal{R}_i}(z) \\ \leftrightarrow & \Phi^{-1}(\underline{u}_{\mathcal{R}_i}(z)) - \frac{\ln 0.8 - \mu_i}{\sigma_i} \leq \varepsilon^{\leq -20}_{i,t} \leq \Phi^{-1}(\bar{u}_{\mathcal{R}_i}(z)) - \frac{\ln 0.8 - \mu_i}{\sigma_i}. \end{aligned} \tag{11}$$

Equations 9-11 and the assumption that $(\varepsilon^{\geq 0}, \varepsilon^{\geq 20}, \varepsilon^{\leq -20}) \sim \mathcal{N}(0, \Sigma)$ allow me to compute

$$\mathbf{P}\left(P_{i,t}^{\geq 0} = x, P_{i,t}^{\geq 20} = y, P_{i,t}^{\leq -20} = z | \left(\mu_i, \sigma_i\right), \mathcal{R}_i\right) \tag{12}$$

as the integral of a normal density over a cube. Since I do not use observations in which the answer to any of the questions is "do not know/refuse," observations where responses for at least one of the question is missing correspond to instances where not all questions were asked. For instance, in all observations before 2008, only $P^{\geq 0}$ was asked. For these observations, the likelihood of the given answers omits the questions that were not asked and it becomes an integral over a real interval (if only one question is asked) or a rectangle (if two questions were asked). With this clarification, I use the same notation in Equation 12 for complete and incomplete sets of answers.

Now, I can write the likelihood of observing an individual i with responses

$$\{(P_{i,t}^{\geq 0}, P_{i,t}^{\geq 20}, P_{i,t}^{\leq -20})\}_{t \in \mathcal{T}(i)}$$

conditional on his rounding type as

$$\ell\left(\left\{(P_{i,t}^{\geq 0}, P_{i,t}^{\geq 20}, P_{i,t}^{\leq -20})\right\}_{t \in \mathcal{T}(i)} | \mathcal{R}_i\right) = \frac{1}{n^2} \sum_{(\mu_i, \sigma_i) \in (\mu, \sigma)^{\#}} \left(\prod_{t \in \mathcal{T}(i)} \mathbf{P}\left(P_{i,t}^{\geq 0}, P_{i,t}^{\geq 20}, P_{i,t}^{\leq -20} | \left(\mu_i, \sigma_i\right), \mathcal{R}_i\right)\right),$$

where I have integrated over the n^2 equiprobable (μ, σ) grid-points. The unconditional likelihood follows from integrating over the rounding types using the prior $\vec{\wp}$,

$$\ell\left(\left\{(P_{i,t}^{\geq 0}, P_{i,t}^{\geq 20}, P_{i,t}^{\leq -20})\right\}_{t \in \mathcal{T}(i)}\right) = \sum_{\mathcal{R}_i \in \{5,10,25,50,100\}} \wp_{\mathcal{R}_i} \times \ell\left(\left\{(P_{i,t}^{\geq 0}, P_{i,t}^{\geq 20}, P_{i,t}^{\leq -20})\right\}_{t \in \mathcal{T}(i)} | \mathcal{R}_i\right)$$

Finally, the log-likelihood function comes from aggregating over individuals

$$\ln \mathcal{L}\left(\left\{\left\{P_{i,t}^{\geq 0}, P_{i,t}^{\geq 20}, P_{i,t}^{\leq -20}\right\}_{t \in \mathcal{T}(i)}\right\}_{i \in \mathcal{I}} | \vartheta^{\mathrm{B}}\right) = \sum_{i \in \mathcal{I}} \ln \ell\left(\left\{(P_{i,t}^{\geq 0}, P_{i,t}^{\geq 20}, P_{i,t}^{\leq -20})\right\}_{t \in \mathcal{T}(i)}\right).$$

A.2 Parameter Estimates

I estimate the beliefs model (1,2) by maximum likelihood for every level of educational attainment,

$$\vartheta_E^B = \arg\max_{\vartheta} \ln \mathcal{L}\left(\left\{\left\{P_{i,t}^{\geq 0}, P_{i,t}^{\geq 20}, P_{i,t}^{\leq -20}\right\}_{t\in\mathcal{T}(i)}\right\}_{i\in\mathcal{I}_E} |\vartheta\right)$$

where E indexes educational attainment levels—less than high school, high school, and college.

Table 9 presents the parameter estimates for every level of educational attainment. Readers should note, however, that the estimates of Equation 2 do not have the traditional mean-covariance interpretation of multivariate normal parameters. The reason is that the actual distributions from which (μ, σ) are drawn condition on the event $\sigma > 0$. These estimates are presented for completeness; for interpreting and comparing the belief distributions that go into the structural model, I refer readers to the depictions of the discretized belief distributions (Figure 5 in the main text).

Table 9: Maximum-likelihood estimates of the beliefs model

	High School	College
ν_μ	−0.108	−0.071
	(0.014)	(0.009)
ν_σ	0.499	0.418
	(0.009)	(0.008)
$\Psi_{1,1}$	0.019	0.016
	(0.001)	(0.001)
$\Psi_{2,1}$	0.011	0.008
	(0.002)	(0.001)
$\Psi_{2,2}$	0.060	0.035
	(0.005)	(0.003)
$\Sigma_{1,1}$	0.607	0.452
	(0.007)	(0.008)
$\Sigma_{2,2}$	0.574	0.460
	(0.010)	(0.011)
$\Sigma_{3,3}$	0.642	0.397
	(0.011)	(0.009)
$\wp 5$	0.403	0.512
	(0.006)	(0.009)
$\wp 10$	0.425	0.399
	(0.006)	(0.009)
$\wp 25$	0.043	0.029
	(0.003)	(0.004)
$\wp 50$	0.116	0.052
	(0.004)	(0.005)
$\wp 100$	0.013	0.007
	(0.002)	(0.002)
Log-Likeligood	−121667.120	−59763.869
N. Obs	24027	11184
N. Individuals	8463	3562
N. Excluded DK/RF Obs.	3216	613

Standard errors come from the inverse of the negative of the hessian of the log-likelihood function, evaluated exactly at the parameter estimates using automatic differentiation tools.

B. Calibration of the Medical-Expenditures Process

The shocks $\texttt{oop}_{i,t}$ represent the ratio of an agent's income that is used up by out-of pocket medical expenditures in a year. To approximate their distribution at different ages and for agents with different levels of educational attainment, I use the RAND HRS longitudinal file, which constructs various variables of interest in a manner that is consistent across HRS waves.

I start by defining a measure of what would be a retiree's household's income. The measure that I use includes the household's earnings, income from pensions and annuities, income from Social Security Disability and Supplemental Security Income, and income from Social Security retirement. This corresponds to the sum of the RAND HRS variables `IEARN`, `IPENA`, `ISSDI`, and `ISRET`. It also corresponds to their measure of "total income" minus government transfers, capital income, and income from "other sources."

Starting with the third wave of the HRS, The RAND HRS longitudinal file includes a measure of out of pocket medical expenses over the previous two years at the time of the interview, `OOPMD`.[37] I divide this measure by two to obtain an estimate of medical expenses over a year at the household level. The ratio between this measure and the previously defined income of the respondent's household—for households with strictly positive income—is what I take as the ratio of out-of-pocket medical-expenditures to income over a year; I denote it with `oop`. Figure 7 depicts the distribution of `oop` for individuals of different ages and levels of education.

To construct discrete distributions that approximate the variability `oop`, I start by grouping observations according to their level of education and age. I use 5-year age bins $[66, 70]$, $[71, 75]$,...,$[85, 90]$ and a final $[91, 100]$ bin. For each combination of age-group and education, I construct discrete equiprobable distributions using quantiles of the empirical distribution of `oop`. First, I split the $[0, 1]$ interval in n intervals of the same length, where n is the number of points of the discrete approximation—in my case, $n = 7$. Then, I take the midpoint of each interval and denote with Q the set of midpoints—for $n = 7$, $Q = \{0.071, 0.214, \ldots, 0.786, 0.929\}$. Finally, for every $q \in Q$, I obtain the q quantile of the empirical distribution of `oop` for the given group. My approximation of the distribution of `oop` is a discrete random variable where the possible draws are the n previously obtained quantiles and each of them occurs with probability $1/n$. Table 10 displays the points that I use in the model for every age group and level of educational attainment.

[37] The measure is not constructed for the first wave. In the second wave, the question that is used to build the measure had a different time horizon and therefore I exclude it.

Table 10: Discrete approximations of medical expenditures/income ratios

Age Group	Equiprobable Points						
High-School							
[50,55]	0.000	0.004	0.010	0.019	0.033	0.064	0.205
(55,60]	0.000	0.005	0.013	0.023	0.040	0.077	0.245
(60,65]	0.000	0.008	0.018	0.032	0.055	0.104	0.290
(65,70]	0.001	0.011	0.023	0.038	0.064	0.111	0.264
(70,75]	0.002	0.014	0.028	0.046	0.074	0.126	0.293
(75,80]	0.001	0.015	0.031	0.053	0.084	0.143	0.346
(80,85]	0.001	0.016	0.033	0.059	0.096	0.168	0.433
(85,90]	0.000	0.016	0.036	0.066	0.110	0.229	0.849
(90,100]	0.000	0.011	0.034	0.069	0.131	0.301	1.479
College							
[50,55]	0.000	0.003	0.007	0.012	0.021	0.039	0.121
(55,60]	0.001	0.005	0.010	0.016	0.027	0.049	0.163
(60,65]	0.002	0.007	0.014	0.023	0.040	0.078	0.227
(65,70]	0.003	0.010	0.019	0.031	0.050	0.089	0.227
(70,75]	0.004	0.013	0.024	0.039	0.060	0.103	0.262
(75,80]	0.004	0.015	0.028	0.047	0.074	0.123	0.294
(80,85]	0.004	0.017	0.033	0.054	0.089	0.155	0.410
(85,90]	0.002	0.015	0.033	0.057	0.100	0.191	0.719
(90,100]	0.000	0.015	0.039	0.075	0.160	0.389	1.485

The table presents approximations to the distribution of health expenditure shocks as a fraction of income. I approximate de distribution of these shocks for each age group and level of education with an equiprobable discrete distribution. Each row displays the seven points used to approximate the distribution for each age group and level of education. Each point has a probability of 1/7. See the text for a description of how I obtain the points.

C. Recursive Formulation and Normalization of the Model

Individual subscripts are dropped for simplicity throughout this section.

An agent starts his life not having paid the stock-market entry cost. In periods when the cost has not been paid, the agent observes his risk-free resources and his permanent income, and then decides whether to enter the stock market or not. His value function is

$$V_t^{\texttt{Out}}(M_t, P_t) = \max\{V_t^{\texttt{Stay}}(M_t, P_t),\ V_t^{\texttt{In}}(M_t - F \times P_t, 0, P_t)\},$$

where $V_t^{\texttt{Stay}}(\cdot)$ is the value function of an agent who stays out of the stock market and $V^{\texttt{In}}(\cdot)$ is the value function of an agent who has already paid the stock-market entry cost.

An agent who has just decided not to pay the stock-market entry cost decides how much to consume out of his assets, knowing that in the next time period he will have the opportunity to enter the stock market again. His value function is

$$V_t^{\texttt{Stay}}(M_t, P_t) = \max_{C_t} u(C_t) + \beta\delta_{t+1}\mathbb{E}_t\left[V_{t+1}^{\texttt{Out}}(M_{t+1}, P_{t+1})\right] + \not\delta_{t+1}\mathbb{B}(A_t)$$

$$\text{Subject to:}$$

$$\begin{aligned} 0 \leq & C_t \leq M_t \\ \\ A_t =& M_t - C_t \\ M_{t+1} =& RA_t + Y_{t+1} \\ P_{t+1} =& \Gamma_{t+1}\psi_{t+1}P_t \\ Y_{t+1} =& \theta_{t+1}P_{t+1} \end{aligned}$$

Finally, an agent who has already paid the stock-market entry cost observes his balances in both the risky and risk-free assets and his permanent income, and then decides how to reallocate his balances and how much to consume. He forms expectations about the future knowing that he will not need to pay the entry cost again. His value function is

$$\begin{aligned} V_t^{\texttt{In}}(M_t, N_t, P_t) = \max_{C_t, D_t} u(C_t) + \beta\delta_{t+1}\mathbb{E}_t \left[V_{t+1}^{\texttt{In}}(M_{t+1}, N_{t+1}, P_{t+1}) \right] \\ + \not{\delta}_{t+1}\mathbb{B}(A_t + \tilde{N}_t) \end{aligned}$$

Subject to:

$$-N_t \leq D_t \leq M_t, \quad 0 \leq C_t \leq \tilde{M}_t$$

$$\begin{aligned} \tilde{M}_t &= M_t - D_t \left(1 - 1_{[D_t \leq 0]}\tau\right) \\ \tilde{N}_t &= N_t + D_t \\ A_t &= \tilde{M}_t - C_t \\ M_{t+1} &= RA_t + Y_{t+1} \\ N_{t+1} &= \tilde{R}_{t+1}\tilde{N}_t \\ P_{t+1} &= \Gamma_{t+1}\psi_{t+1}P_t \\ Y_{t+1} &= \theta_{t+1}P_{t+1} \end{aligned} .$$

I assume that the utility function $u(\cdot)$ and the bequest function $\mathbb{B}(\cdot)$ are homothetic of the same degree $(1-\rho)$. With this assumption, the problem can be normalized by permanent income, following Carroll (2022). Using lower case variables to denote their upper-case counterparts normalized by permanent income ($x_t = X_t/P_t$) and defining $\tilde{\Gamma}_t = \Gamma_t\psi_t$, we can write normalized versions of the previous value functions as

$$v_t^{\texttt{Out}}(m_t) = \max\{v_t^{\texttt{Stay}}(m_t), v_t^{\texttt{In}}(m_t - F, 0)\}, \tag{13}$$

$$\begin{aligned} v_t^{\texttt{Stay}}(m_t) = \max_{c_t} u(c_t) + \beta\delta_{t+1}\mathbb{E}_t \left[\tilde{\Gamma}_{t+1}^{1-\rho} v_{t+1}^{\texttt{Out}}(m_{t+1}) \right] + \not{\delta}_{t+1}\mathbb{B}(a_t) \\ \text{Subject to:} \\ 0 \leq c_t \leq m_t \\ a_t = m_t - c_t \\ m_{t+1} = \frac{R}{\tilde{\Gamma}_{t+1}} a_t + \theta_{t+1} \end{aligned} , \tag{14}$$

and

$$v_t^{\mathtt{In}}(m_t, n_t) = \max_{c_t, d_t} u(c_t) + \beta\delta_{t+1}\mathbb{E}_t\left[\tilde{\Gamma}_{t+1}^{1-\rho} v_{t+1}^{\mathtt{In}}\left(m_{t+1}, n_{t+1}\right)\right] + \not{\delta}_{t+1}\mathbb{B}(a_t + \tilde{n}_t)$$

Subject to:

$$-n_t \leq d_t \leq m_t, \quad 0 \leq c_t \leq \tilde{m}_t$$

$$\begin{aligned}
\tilde{m}_t =& m_t - d_t\left(1 - 1_{[d_t \leq 0]}\tau\right) \\
\tilde{n}_t =& n_t + d_t \\
a_t =& \tilde{m}_t - c_t \\
m_{t+1} =& \frac{R}{\tilde{\Gamma}_{t+1}} a_t + \theta_{t+1} \\
n_{t+1} =& \frac{\tilde{R}_{t+1}}{\tilde{\Gamma}_{t+1}} \tilde{n}_t
\end{aligned}$$

It can be shown that

$$\begin{aligned}
V_t^{\mathtt{Out}}(M_t, P_t) =& P_t^{1-\rho} v_t^{\mathtt{Out}}(m_t), \\
V_t^{\mathtt{Stay}}(M_t, P_t) =& P_t^{1-\rho} v_t^{\mathtt{Stay}}(m_t), \\
V_t^{\mathtt{In}}(M_t, N_t, P_t) =& P_t^{1-\rho} v_t^{\mathtt{In}}(m_t, n_t)
\end{aligned}$$

and that the policy functions that solve each of the problems are related through

$$\begin{aligned}
C_t^{\mathtt{Stay}}(M_t, P_t) &= P_t c_t^{\mathtt{Stay}}(m_t) \\
C_t^{\mathtt{In}}(M_t, N_t, P_t) &= P_t c_t^{\mathtt{In}}(m_t, n_t) \\
D_t(M_t, N_t, P_t) &= P_t d_t(m_t, n_t).
\end{aligned}$$

Therefore, I solve the normalized problem and re-scale its solutions to obtain the original problem's solutions.

C.1 Partition Into Stages

An additional insight that facilitates solving the dynamic problem of the agent who has paid the stock-market entry cost is that the two decisions that he takes in a period (rebalancing his assets and consuming) can be seen as happening sequentially. This is convenient because the sequential sub-problems are easier to solve than the multi-choice full problem.

To re-express the problem, I take the order of the decisions to be: first rebalance assets, then consume. I denote the stages at which these decisions are taken with `Reb` and `Cns`. I will use $v^{\texttt{Reb}}(\cdot)$ and $v^{\texttt{Cns}}(\cdot)$ to represent the respective *stage value functions*.

I now present each stage in detail, working backwards in time.

C.1.1 *Consumption stage,* `Cns`

The important fact to realize at this stage is that the first thing that the agent will do in period $t+1$ is make his asset-rebalancing decision. Therefore, that is the value function about which the agent forms expectations.

The consumption stage problem is

$$v_t^{\texttt{Cns}}(\tilde{m}_t, \tilde{n}_t) = \max_{c_t} u(c_t) + \beta \delta_{t+1} \mathbb{E}_t \left[\tilde{\Gamma}_{t+1}^{1-\rho} v_{t+1}^{\texttt{Reb}}(m_{t+1}, n_{t+1}) \right] + \not{\delta}_{t+1} \mathbb{B}(a_t + \tilde{n}_t)$$

$$\begin{aligned} &\text{Subject to:} \\ 0 &\leq c_t \leq \tilde{m}_t \\ a_t &= \tilde{m}_t - c_t \\ m_{t+1} &= \frac{R}{\tilde{\Gamma}_{t+1}} a_t + \theta_{t+1} \\ n_{t+1} &= \frac{\tilde{R}_{t+1}}{\tilde{\Gamma}_{t+1}} \tilde{n}_t \end{aligned} \tag{15}$$

C.1.2 *Rebalancing stage,* `Reb`

The first decision that an agent takes is how to reallocate his assets. His payoff is given by the subsequent consumption problem's value function, evaluated at his post-rebalancing assets.

$$v_t^{\texttt{Reb}}(m_t, n_t) = \max_{d_t} v_t^{\texttt{Cns}}(\tilde{m}_t, \tilde{n}_t)$$

$$\begin{aligned} &\text{Subject to:} \\ -n_t &\leq d_t \leq m_t \\ \tilde{m}_t &= m_t - d_t \left(1 - 1_{[d_t \leq 0]} \tau\right) \\ \tilde{n}_t &= n_t + d_t \end{aligned} \tag{16}$$

D. First Order Conditions and Value-Function Derivatives

The computational solution of the model uses the first-order conditions of the optimization problems and the derivatives of the value functions defined above. This appendix writes the first-order conditions and value-function derivatives explicitly.

D.0.1 *Agent who is staying out of the stock market,* `Stay`

The first order condition of the maximization problem in Equation 14 is

$$u'(c_t) = \beta R \delta_{t+1} \mathbb{E}_t \left[\tilde{\Gamma}_{t+1}^{-\rho} \frac{\partial v_{t+1}^{\texttt{Out}}}{\partial m_{t+1}} \right] + \not{\delta}_{t+1} \mathbb{B}'(a_t). \tag{17}$$

The condition, while necessary, is not sufficient because $v^{\texttt{Out}}(\cdot)$ is not concave. Therefore, I use the DC-EGM method (Iskhakov et al. 2017) to solve this sub-problem.

D.0.2 *Consumption stage,* `Cns`

The first order condition for an interior solution ($c < \tilde{m}$) of the consumption stage problem (Equation 15) is

$$u'(c_t) = \beta R \delta_{t+1} \mathbb{E}_t \left[\tilde{\Gamma}_{t+1}^{-\rho} \frac{\partial v_{t+1}^{\texttt{Reb}}}{\partial m_{t+1}} \right] + \not{\delta}_{t+1} \mathbb{B}'(a_t + \tilde{n}_t) \tag{18}$$

The derivatives of the stage value function are

$$\frac{\partial v_t^{\texttt{Cns}}(\tilde{m}_t, \tilde{n}_t)}{\partial \tilde{m}_t} = u'(c_t) \tag{19}$$

$$\frac{\partial v_t^{\texttt{Cns}}(\tilde{m}_t, \tilde{n}_t)}{\partial \tilde{n}_t} = \beta \delta_{t+1} \mathbb{E}_t \left[\tilde{R}_{t+1} \tilde{\Gamma}_{t+1}^{-\rho} \frac{\partial v_{t+1}^{\texttt{Reb}}}{\partial n_{t+1}} \right] + \not{\delta}_{t+1} \mathbb{B}'(a_t + \tilde{n}_t) \tag{20}$$

D.0.3 *Rebalancing stage,* `Reb`

The first order condition for a solution of the type $d \in [(-n, 0) \cup (0, m)]$ in the rebalancing stage problem (Equation 16) is

$$\left(1 - 1_{[d_t \leq 0]} \tau\right) \frac{\partial v_t^{\texttt{Cns}}}{\partial \tilde{m}_t} = \frac{\partial v_t^{\texttt{Cns}}}{\partial \tilde{n}_t}, \tag{21}$$

and a necessary condition for a solution of the type $d = 0$ is

$$(1-\tau)\,\frac{\partial v_t^{\text{Cns}}}{\partial \tilde{m}_t} \leq \frac{\partial v_t^{\text{Cns}}}{\partial \tilde{n}_t} \leq \frac{\partial v_t^{\text{Cns}}}{\partial \tilde{m}_t} \tag{22}$$

The derivatives of the stage value function are

$$\frac{\partial v_t^{\text{Reb}}(m_t, n_t)}{\partial m_t} = \max\left\{\frac{\partial v_t^{\text{Cns}}}{\partial \tilde{m}}, \frac{\partial v_t^{\text{Cns}}}{\partial \tilde{n}}\right\} \tag{23}$$

$$\frac{\partial v_t^{\text{Reb}}(m_t, n_t)}{\partial n_t} = \max\left\{(1-\tau)\frac{\partial v_t^{\text{Cns}}}{\partial \tilde{m}_t}, \frac{\partial v_t^{\text{Cns}}}{\partial \tilde{n}_t}\right\} \tag{24}$$

Table 11: Grids and Discretizations

	Symbol	# Points	Type of grid/Discretization
Grids for $a, m, n, \tilde{m}, \tilde{n}$	$a^{\#}$, $m^{\#}$, $n^{\#}$, $\tilde{m}^{\#}$, $\tilde{n}^{\#}$	101	Equispaced in logs between $1e-6$ and $5e3$, with 0 added.
Perm. Inc. Shock	ψ	5	Equiprobable.
Trans. Inc. Shock	$\tilde{\theta}$	5	Equiprobable.
Risky return	$\tilde{R}$	5	Equiprobable.

E. Numerical Solution of the Life-Cycle Model.

E.1 Grids and discretizations

The solution of the model uses various discrete grids over state variables and discretizations of stochastic variables. Table 11 summarizes the grids and discretization schemes that I use for every variable and shock. The only shock discretization not addressed in Table 11 is the out-of-pocket medical expenditure shock, which I discuss in detail in Appendix B.

E.2 Transformed-space interpolation

In my solution, I treat the continuous choice variables c and d as continuous, instead of discretizing them. Because of this decision, and the multiple shocks in the model, I must evaluate value functions and their derivatives on values of the state vector that are not on my grids. In these instances, I interpolate (and extrapolate) using on-grid values.

To improve my approximation of properties of the value and marginal-value functions, such as their curvature, and the fact that they approach $-\infty$ or ∞ as wealth approaches zero, I perform my interpolations and extrapolations in a "transformed" space. The trick, discussed in Carroll (2022), consists in finding a transformation $T : \mathbb{R} \to \mathbb{R}$ such that $T(f(\cdot))$ behaves more like an affine function than $f(\cdot)$, the function that we are trying to approximate. Then, with our chosen transformation, we create an interpolator $\hat{g}(\cdot)$ for $T(f(\cdot))$. When asked to approximate $f(x)$ for some off-grid x we return $T^{-1}(\hat{g}(x))$, where $T^{-1}(\cdot)$ is the inverse of $T(\cdot)$.

I apply this trick when constructing interpolators for value functions and marginal value functions. For value functions, I use

$$T(x) = u^{-1}(x) = ((1-\rho)\times x)^{\frac{1}{1-\rho}}, \qquad T^{-1}(x) = u(x) = \frac{x^{1-\rho}}{1-\rho}.$$

For marginal value functions, I use

$$T(x) = u'^{-1}(x) = x^{-\frac{1}{\rho}}, \qquad T^{-1}(x) = u'(x) = x^{-\rho}.$$

E.3 Solving the Consumption Stage, Cns

For this stage, I use the method of endogenous gridpoints (Carroll 2006) over risk-free resources at different fixed levels of risky resources.

First, for every $\tilde{n}$ in the risky-asset balances grid $\tilde{n}^{\#}$,

- Apply the endogenous gridpoint method using Equation 18 over the grid $a^{\#}$ for end-of-period risk-free assets. The result is a set of optimal consumption points on an endogenous grid of post-rebalancing risk-free assets $\tilde{m}_t^{\#-\text{endog}}(\tilde{n})$,

$$c_t^*(\tilde{m},\tilde{n}) \text{ for } \tilde{m} \in \tilde{m}_t^{\#-\text{endog}}(\tilde{n}).$$

- Denote the endogenous $\tilde{m}$ associated with $a_t = 0$ by $\tilde{m}_0(\tilde{n})$. This is the point where the liquidity constraint stops binding. If $\tilde{m}_0(\tilde{n}) > 0$, then add $\tilde{m} = 0$, $c_t^*(0,\tilde{n}) = 0$ to the set of endogenous risk-free assets and optimal consumption points.
- Use the optimal consumption points to find v_t^{Cns}, $\frac{\partial v_t^{\text{Cns}}}{\partial \tilde{m}}$, $\frac{\partial v_t^{\text{Cns}}}{\partial \tilde{n}}$ at $(\tilde{m},\tilde{n})$, for $\tilde{m} \in \tilde{m}_t^{\#-\text{endog}}(\tilde{n})$ using Equations 15, 19, and 20.

The result is a set of $(\tilde{m},\tilde{n})$ points on which we know the consumption, value, and marginal value functions. This set of points is not a rectangular grid because the values of $\tilde{m}$ are different for every $\tilde{n}$. The next step is to use the current points to obtain an approximation of the functions over a rectangular grid.

I start with an exogenous grid for post-rebalancing risk-free assets $\tilde{m}^{\#}$ which I augment by adding the points where the liquidity constraint stops binding, $\{\tilde{m}_0(\tilde{n}) : \tilde{n} \in \tilde{n}^{\#}\}$. Denote the augmented grid with $\tilde{m}^{\#+}$. For every $\tilde{n}$ in the risky-asset balances grid $\tilde{n}^{\#}$,

- Use the values of c_t^*, v_t^{Cns}, $\frac{\partial v_t^{\text{Cns}}}{\partial \tilde{m}}$, $\frac{\partial v_t^{\text{Cns}}}{\partial \tilde{n}}$ calculated at $\{(\tilde{m},\tilde{n}) : \tilde{m} \in \tilde{m}_t^{\#-\text{endog}}(\tilde{n})\}$ to approximate the value of c^*, v_t^{Cns}, $\frac{\partial v_t^{\text{Cns}}}{\partial \tilde{m}}$, $\frac{\partial v_t^{\text{Cns}}}{\partial \tilde{n}}$ at $\{(\tilde{m},\tilde{n}) : \tilde{m} \in \tilde{m}^{\#+}\}$ using transformed-space linear interpolation (and extrapolation).

This process yields approximations of the functions of interest on a rectangular grid, $\{(\tilde{m},\tilde{n}) : \tilde{m} \in \tilde{m}^{\#+} \text{ and } \tilde{n} \in \tilde{n}^{\#}\}$. I use these approximations to construct bilinear transformed-space interpolators for c_t^*, $v_t^{\texttt{Cns}}$, $\frac{\partial v_t^{\texttt{Cns}}}{\partial \tilde{m}}$, $\frac{\partial v_t^{\texttt{Cns}}}{\partial \tilde{n}}$.

E.4 Solving the Rebalancing Stage, Reb

In this stage, I look for the optimal deposit/withdrawal function $d_t^*(m,n)$.

I start by defining the following convenient transformation of the optimal deposit/withdrawal

$$\mathfrak{d}_t(m,n) = \begin{cases} d_t^*(m,n)/m & d_t^*(m,n) > 0 \\ d_t^*(m,n)/n & d_t^*(m,n) < 0 \\ 0 & d_t^*(m,n) = 0. \end{cases}$$

The transformation simply re-scales the deposits or withdrawals by the balance of the fund that they are coming from, so that $\mathfrak{d}_t = 1$ corresponds to moving all the risk-free balances to the risky stocks fund, and $\mathfrak{d}_t = -1$ corresponds to withdrawing all balances from the stocks fund.

I search for the optimal $\mathfrak{d}_t$ in a rectangular exogenous grid $\{(m,n) : m \in m^{\#} \text{ and } n \in n^{\#}\}$. The search uses the first order conditions in Equations 21 and 22 and proceeds as follows.

For every (m,n) in the rectangular grid,

- Evaluate $\frac{\partial v_t^{\texttt{Reb}}(m,n)}{\partial \tilde{m}}$ and $\frac{\partial v_t^{\texttt{Reb}}(m,n)}{\partial \tilde{n}}$.
- If $(1-\tau)\frac{\partial v_t^{\texttt{Cns}}(m,n)}{\partial \tilde{m}} \leq \frac{\partial v_t^{\texttt{Cns}}(m,n)}{\partial \tilde{n}} \leq \frac{\partial v_t^{\texttt{Cns}}(m,n)}{\partial \tilde{m}}$, then $\mathfrak{d}_t(m,n) = 0$.
- If $\frac{\partial v_t^{\texttt{Cns}}(m,n)}{\partial \tilde{n}} < (1-\tau)\frac{\partial v_t^{\texttt{Cns}}(m,n)}{\partial \tilde{m}}$
  - We know that the solution involves withdrawing funds, $\mathfrak{d}_t(m,n) < 0$. We have to check the corner solution $\mathfrak{d}_t(m,n) = -1$.
  - If
  $$\frac{\partial v_t^{\texttt{Cns}}(m+(1-\tau)n,0)}{\partial \tilde{n}} < (1-\tau)\frac{\partial v_t^{\texttt{Cns}}(m+(1-\tau)n,0)}{\partial \tilde{m}},$$
  then set $\mathfrak{d}_t(m,n) = -1$.
  - Otherwise, use bisection search to find the $d^* \in (-1,0)$ that solves
  $$\frac{\partial v_t^{\texttt{Cns}}(m-(1-\tau)d^*,n+d^*)}{\partial \tilde{n}} = (1-\tau)\frac{\partial v_t^{\texttt{Cns}}(m-(1-\tau)d^*,n+d^*)}{\partial \tilde{m}}$$
  and set $\mathfrak{d}_t(m,n) = d^*/n$.

- If $\frac{\partial v_t^{\texttt{Cns}}(m,n)}{\partial \tilde{m}} < \frac{\partial v_t^{\texttt{Cns}}(m,n)}{\partial \tilde{n}}$
  - We know that the solution involves depositing funds, $\mathfrak{d}_t(m,n) > 0$. We also know that the corner solution $\mathfrak{d}_t(m,n) = 1$ is not optimal because it leaves the agent without funds to consume.
  - Use bisection search to find the $d^* \in (0,1)$ that solves

$$\frac{\partial v_t^{\texttt{Cns}}(m - d^*, n + d^*)}{\partial \tilde{m}} = \frac{\partial v_t^{\texttt{Cns}}(m - d^*, n + d^*)}{\partial \tilde{n}}$$

  - Set $\mathfrak{d}_t(m,n) = d^*/m$.

The result of this process is a rectangular grid of asset-combinations and their associated optimal rebalancing solutions. I use these points to construct a bilinear interpolator for $\mathfrak{d}_t(\cdot,\cdot)$. Then, I use the fact that (from Equation 16),

$$\begin{aligned}
v_t^{\texttt{Reb}}(m_t, n_t) =& v_t^{\texttt{Cns}}(\tilde{m}_t, \tilde{n}_t) \\
&\text{Where:} \\
&\quad d_t = \begin{cases} \mathfrak{d}_t(m,n) \times m_t, & \text{If } \mathfrak{d}_t(m,n) \geq 0 \\ \mathfrak{d}_t(m,n) \times n_t, & \text{If } \mathfrak{d}_t(m,n) < 0 \end{cases} \\
&\quad \tilde{m}_t = m_t - d_t \left(1 - 1_{[d_t \leq 0]}\tau\right) \\
&\quad \tilde{n}_t = n_t + d_t
\end{aligned}$$

to calculate $v_t^{\texttt{Reb}}(\cdot,\cdot)$ and its derivatives whenever they are needed.

E.5 Solving the Problem of the Agent Staying Out, `Stay`

Agents who have not paid the one-time entry cost must decide whether to pay it at the start of every period. They do this by comparing the value of staying out and not paying, versus entering and paying, as shown in Equation 13. An agent who enters passes onto the asset-rebalancing stage, `Reb`. An agent who does not enter must choose his consumption knowing that next period he will start outside of the stocks-fund again. I use the "DC-EGM" method (Iskhakov et al. 2017) to solve this problem.

I start with an exogenous grid for end-of-period risk-free assets, $a^{\#}$. I apply the endogenous-gridpoint method inversion over $a^{\#}$ using the first-order condition in Equation 17. The result is a set of candidate endogenous consumption and beginning-of-period-assets points, associated with the exogenous end-of-period-assets points,

$$\left\{ \left(c_t^e(a), m_t^e(a) \right) : \; a \in a^{\#} \right\}.$$

As argued by Iskhakov et al. (2017), these points will not necessarily be optimal. The future discrete decision of whether to pay the cost or not makes the value function not-concave and therefore points that satisfy the first order condition are not necessarily optimal. I calculate the discounted utility associated with the points from the endogenous-gridpoint inversion,

$$v_t^e(a) = u(c_t^e(a)) + \beta \delta_{t+1} \mathbb{E}_t \left[\tilde{\Gamma}_{t+1}^{1-\rho} v_{t+1}^{\texttt{Out}}(a) \right] + \not{\delta}_{t+1} \mathbb{B}(a_t)$$
$$m_{t+1} = \frac{R}{\tilde{\Gamma}_{t+1}} a + \theta_{t+1}.$$

Then, I apply the upper-envelope algorithm in Iskhakov et al. (2017) to the candidate points $\left\{ \left(c_t^e(a), m_t^e(a), v_t^e(a) \right) : \; a \in a^{\#} \right\}$ to eliminate non-optimal points. I add the "kink points" of the value function to the grid. The result is a set of optimal consumption and value points over a refined endogenous grid for start-of-period assets $m_t^{*\#}$, $\left\{ \left(c_t^*(m), v_t^*(m) \right) : \; m \in m^{*\#} \right\}$. I use these points to create linear transformed-space interpolators for $v_t^{\texttt{Stay}}$, $c_t^{\texttt{Stay}}$, and $\frac{\partial v_t^{\texttt{Stay}}}{\partial m}$.

F. Surrogate Bootstrap

The estimated parameters of the structural life-cycle models are functions of the targeted moments. Uncertainty about these moments generates uncertainty about the best-fitting values of the parameters. I quantify this uncertainty by calculating the targeted moments on bootstrapped samples, estimating an approximate—"surrogate"—model on each set of moments, and presenting various summary statistics of the resulting distribution of estimated parameters.

The bootstrapped targeted moments come from education-specific re-samplings of the SCF analytical sample defined in Section 4.1. I divide the sample into high-school and college graduates, and draw 500 bootstrapped samples for each level of education using the re-scaled survey weights. Then, I calculate the targeted moments on each of the bootstrapped samples. This results in 500 vectors of 45 targeted moments each for both levels of education, $\{m_{b,k}^{\text{HS}}\}_{k=0}^{500}$ and $\{m_{b,k}^{\text{College}}\}_{k=0}^{500}$.

To find the estimated parameters that would be result from each vector of moments, I use accurate surrogate models that approximate the relationship between parameters and moments embedded in the true structural models. Estimating the structural models 500 times for each level of education would come at a high computational arising mainly from their solution and simulation at each candidate vector of parameters. Recent studies like Chen, Didisheim, and Scheidegger (2021) and Catherine et al. (2022) show that these costly evaluations can be avoided using accurate approximations of the structural model that can be constructed using known parameter-moments pairs. These approximations are know as "surrogate models." Denoting with $\mathbb{\Theta}$ the space of admissible parameter values and with $\mathbb{M}$ the set of possible values for targeted moments, a model is a function $f : \mathbb{\Theta} \to \mathbb{M}$ and a surrogate model is a different function $\hat{f} : \mathbb{\Theta} \to \mathbb{M}$ that approximates the true model f, but which is ideally much faster to evaluate.

The surrogate models that I use to approximate the true structural models are deep neural networks. The networks have 3 inputs (the parameters $\{\rho, \beta, F\}$) and 45 outputs (the targeted moments), and 3 hidden layers with 128 neurons each. I use sigmoid-linear-unit "SiLU" activation functions for the hidden layers; for the output layer, I use "softplus" functions for positive moments (like the wealth ratio) and sigmoid functions for moments that are shares (like conditional stock-shares and participation rates). I use a different network for each combination of educational attainment and specification of beliefs (High-School, College, and F.I.R.E., Est. Beliefs).

As suggested by Catherine et al. (2022), I train and validate the surrogate models using the parameter-moment points that I evaluate when estimating the true models. The

Table 12: Root-Mean-Squared-Errors of Surrogate Models over Targeted Moments

	High-School		College	
Sample	F.I.R.E.	Est. Beliefs	F.I.R.E.	Est. Beliefs
Training	6.12×10^{-3}	2.56×10^{-3}	4.28×10^{-3}	3.62×10^{-3}
Validation	3.03×10^{-2}	2.08×10^{-2}	1.76×10^{-2}	4.85×10^{-3}

optimization routine outlined in Section 4.2 evaluates each structural model at at least 2,500 points of the space of admissible parameter values Θ. The initial 2,500 points come from a Sobol sequence that covers Θ well. The local optimization runs of the `TikTak` algorithm generate additional evaluations, which concentrate around the best-fitting parameter values. I save the parameter-moments pair of every one of these evaluations for each specification of the model. Then, I randomly split the points into 85% training and 15% validation samples. I train the deep networks using the "Adam" algorithm (Kingma and Ba 2017) to minimize the root-mean-squared-error (RMSE) over the targeted moments. Table 12 presents the RMSEs of every surrogate model at the end of estimation, confirming that they do a good job of approximating the predictions of the true models both in- and out-of-sample.

For each vector of bootstrapped moments, I find the input parameters of its corresponding surrogate model that minimizes the SMM loss function using the same optimization routine that I use for the main estimates. The $k-th$ set of bootstrapped parameter estimates for the pair of education and beliefs-specification $(e, \mathbf{b})$ is

$$\hat{\vartheta}_k^{e,\mathbf{b}} = \arg\min_{\vartheta} \left(m_{b,k}^e - \hat{f}^{e,\mathbf{b}}(\vartheta)\right)' \mathbb{W}^e \left(m_{b,k}^e - \hat{f}^{e,\mathbf{b}}(\vartheta)\right), \tag{25}$$

where $\hat{f}^{e,\mathbf{b}}$ is the surrogate model for education level e and belief specification $\mathbb{b}$. The results are sets of bootstrapped estimates

$$\{\hat{\vartheta}_{b,k}^{\text{HS,F.I.R.E.}}\}_{k=0}^{500}, \{\hat{\vartheta}_{b,k}^{\text{College, F.I.R.E.}}\}_{k=0}^{500}, \text{ and } \{\hat{\vartheta}_{b,k}^{\text{HS, Est. Beliefs}}\}_{k=0}^{500}, \{\hat{\vartheta}_{b,k}^{\text{College, Est. Beliefs}}\}_{k=0}^{500}.$$

I additionally store and present the values of the loss functions associated with each parameter estimate.

Table 13 presents the 5th, 50th, and 95th percentiles of every parameter for every combination of educational attainment and belief specification, in addition to the minimized loss function "MSM Loss." The table shows that all the parameter estimates and attained

Table 13: Different Percentiles of Bootstrapped Estimates

	Percentiles						
	F.I.R.E.			Est. Belifs			Difference
Parameter	P_5	P_{50}	P_{95}	P_5	P_{50}	P_{95}	Frac. > 0
High School							
CRRA (ρ)	8.55	8.60	8.64	4.20	4.22	4.25	1
Disc. Fac (β)	0.32	0.33	0.34	0.75	0.76	0.77	0
Entry Cost ($F \times 100$)	3.04	3.19	3.36	2.34	2.52	2.66	1
MSM Loss	15.44	15.98	16.66	3.67	4.04	4.46	1
College							
CRRA (ρ)	11.39	11.46	11.52	5.06	5.09	5.12	1
Disc. Fac (β)	0.62	0.63	0.64	0.88	0.89	0.89	0
Entry Cost ($F \times 100$)	0.32	0.37	1.64	0.00	0.00	0.00	1
MSM Loss	4.55	5.32	6.29	2.24	2.96	3.86	1

losses are tightly distributed around the main estimates reported in Table 8. For each vector of bootstrapped moments $m^e_{b,k}$, I find the difference between the parameter estimates and attained losses under the F.I.R.E. and "Est. Beliefs" specifications. The last column in Table 13 presents the fraction of moment vectors for which this difference is positive. This column shows the robustness of the conclusions that, for both levels of education, models that use the estimated beliefs improve upon the fit of F.I.R.E. models and do so with lower levels of relative-risk aversion, higher discount factors, and lower entry-costs. These conclusions are true for each of the 500 bootstrapped vectors of moments for each level of education.

The 95% confidence intervals presented in Table 8 correspond to the 2.5-th and 97.5-th percentiles of the bootstrapped values of each parameter, for each model specification.

G. The expected welfare function

The welfare calculations presented in Section 6 rely on the calculation of individuals' objectively expected welfare, which can differ from their subjective expectations due to their misspecified beliefs. This section defines my measure of expected welfare and discusses how I calculate it.

For a given set of beliefs about risky returns denoted with $\mathcal{B} = (\mu, \sigma)$ and other parameters, I solve the life-cycle model and its components described in Appendix C. Denote the resulting policy functions for every age t with $C_t^{\texttt{Stay}}(\cdot; \mathcal{B})$, $D_t(\cdot; \mathcal{B})$, and $C_t^{\texttt{In}}(\cdot; \mathcal{B})$. I calculate functions $\mathfrak{V}_t(\cdot)$ that allow me to find the expected lifetime welfare that an objective observer would expect an agent to derive from his remaining years of life if he behaved according to the policy functions associated with his beliefs $\mathcal{B}$. These functions are

$$
\begin{gathered}
\mathfrak{V}_t^{\texttt{Stay}}(M_t, P_t; \mathcal{B}) = u(C_t) + \beta\delta_{t+1}\mathbb{E}_t\left[\mathfrak{V}_{t+1}^{\texttt{Out}}(M_{t+1}, P_{t+1}; \mathcal{B})\right] + \not{\delta}_{t+1}\mathbb{B}(A_t) \\
\text{Where:} \\
C_t = C_t^{\texttt{Stay}}(M_t, P_t; \mathcal{B}) \\
A_t = M_t - C_t, \quad M_{t+1} = RA_t + Y_{t+1}, \quad P_{t+1} = \Gamma_{t+1}\psi_{t+1}P_t, \quad Y_{t+1} = \theta_{t+1}P_{t+1}
\end{gathered} \tag{26}
$$

for agents who have not paid the risky-asset entry cost and decide to not pay it in t, and

$$
\begin{gathered}
\mathfrak{V}_t^{\texttt{In}}(M_t, N_t, P_t; \mathcal{B}) = u(C_t) + \beta\delta_{t+1}\mathbb{E}_t\left[\mathfrak{V}_{t+1}^{\texttt{In}}(M_{t+1}, N_{t+1}, P_{t+1}; \mathcal{B})\right] \\
+ \not{\delta}_{t+1}\mathbb{B}(A_t + N_t + D_t) \\
\text{Where:} \\
D_t = D_t(M_t, N_t, P_t; \mathcal{B}), \quad C_t = C_t^{\texttt{In}}(M_t, N_t, P_t; \mathcal{B}) \\
A_t = M_t - D_t\left(1 - 1_{[D_t \leq 0]}\tau\right) - C_t, \quad M_{t+1} = RA_t + Y_{t+1} \\
N_{t+1} = \tilde{R}_{t+1} \times (N_t + D_t), \quad P_{t+1} = \Gamma_{t+1}\psi_{t+1}P_t, \quad Y_{t+1} = \theta_{t+1}P_{t+1}
\end{gathered} \tag{27}
$$

for agents who have already paid the entry cost. Equations 26 and 27 differ from the value functions defined in Appendix C because the expectations are taken using the true distribution of risky-asset returns. Numerically, I construct interpolators for these functions iterating backwards, using the solved policy functions and the same grids and discretizations described in Appendix E.

The simplified notation that I use for $\mathfrak{V}_t(\cdot)$ in the main text is corresponds to

$$\mathfrak{V}_t(P_t, M_t, N_t = 0, \texttt{Paid}_t = 0; \mathcal{B}) \equiv \max \left\{ \mathfrak{V}_t^{\texttt{Stay}}(M_t, P_t; \mathcal{B}), \mathfrak{V}_t^{\texttt{In}}(M_t - F \times P_t, 0, P_t; \mathcal{B}) \right\}$$

$$\mathfrak{V}_t(P_t, M_t, N_t, \texttt{Paid}_t = 1; \mathcal{B}) \equiv \mathfrak{V}_t^{\texttt{In}}(M_t, N_t, P_t; \mathcal{B}).$$